Master course "A Medley of Advanced Probability" – Autumn 2020
Exercise sheet 1 — Solution (Lévy processes)

Notation: Recall that for each process $X = (X_t)_{t\geq 0}$ and time $\tau \geq 0$, we define the shifted process $X^{(\tau)}$ by $X_t^{(\tau)} = X_{\tau+t} - X_\tau$. When $S = (S_t)_{t\geq 0}$ satisfies that $t \mapsto S_t$ is non-decreasing on $\mathbb{R}_{\geq 0}$ almost surely, let $X\lfloor_S$ denote the process $(X_{S_t})_{t\geq 0}$. We denote by $X \overset{(d)}{=} Y$ the fact that the random variables (including processes) X and Y have the same law.

Exercise 1 (Warm-up)**.** Consider a random time $T \geq 0$ and two random processes $S = (S_t)_{t\geq 0}$ and $X = (X_s)_{s\geq 0}$ such that S is almost surely non-decreasing.

(1) Assume that the triplet (T, S, X) is independent, and let $(\tilde{T}, \tilde{S}, \tilde{X})$ be an independent copy of it. Prove that we have $X^{(T)} \overset{(d)}{=} \tilde{X}^{(\tilde{T})}$ and $X\lfloor_S \overset{(d)}{=} \tilde{X}\lfloor_{\tilde{S}}$.

(2) Assume that X has almost surely càdlàg paths. Let $\sigma(X, T)$ denote the σ-field generated by (X, T), completed with sets of probability zero. Show that the process $X^{(T)}$ is measurable in $\sigma(X, T)$, and similarly, show that the process $X\lfloor_S$ is measurable in $\sigma(X, S)$.

Solution of Exercise 1. This exercise explores the following question: When are the laws of the processes $X^{(T)}$ and $X\lfloor_S$ determined by the laws of T, S and X? Since the law of a random process is determined by its marginal distribution at finitely many times, the question basically boils down to: When is the law of X_T determined by those of X and T? It seems that the answer is trivially yes, since X_T is a function of X and T. But the following example shows otherwise:

Let τ be a random time whose law has no atom (i.e. $\mathbb{P}(\tau = t) = 0$ for any fixed t), and define two processes X and $\tilde{X}$ by $X_t = 0$ and $\tilde{X}_t = \mathbb{1}_{\{t=\tau\}}$ for all $t \geq 0$. For any $t_1, \ldots, t_p \geq 0$, since $\mathbb{P}(\tau \in \{t_1, \ldots, t_p\}) = 0$, we have $(\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_p}, \tau) = (0, \ldots, 0, \tau) = (X_{t_1}, \ldots, X_{t_p}, \tau)$ almost surely. Therefore the pairs (X, τ) and $(\tilde{X}, \tau)$ have the same law. However, we have $X_\tau = 0$ and $\tilde{X}_\tau = 1$ almost surely.

This example shows that X_τ may not be a measurable function of the pair (X, τ), since otherwise its law would be determined by the law of (X, τ). The problem here is that X_τ depends on the value of X_t for uncountably many t, hence its law cannot be determined by the law of X (which captures only the joint distributions of X_t for countably many t's) without some regularity assumption on the process X. This is reminiscent to the fact that a càdlàg process may have the same law as a process which is never càdlàg. (See the discussion on càdlàg versions of stochastic processes in the background material of the course.) The exercise proposes two sufficient conditions which ensure that the laws of $X^{(T)}$ and $X\lfloor_S$ are determined by those of T, X and S: either (1) T, X and S are mutually independent, or (2) the process X has almost surely càdlàg paths.

(1) The intuition for this question is that if two random variables U, V are independent, then for any function $f(U, V)$ measurable with respect to U, we have $\mathbb{E}[f(U, V)|V] = g(V)$ with $g(v) := E[f(U, v)]$. If g is also measurable, then we can define $\mathbb{E}[f(U, V)] := \mathbb{E}[g(V)]$, without ever requiring $f(U, V)$ to be jointly measurable with respect to (U, V).

Concretely, fix a sequence $t_1, \ldots, t_p \geq 0$ and a bounded measurable function $f : \mathbb{R}^p \to \mathbb{R}$. Let $g : \mathbb{R}^p \to \mathbb{R}$ be defined by $g(s_1, \ldots, s_p) = \mathbb{E}[f(X_{s_1}, \ldots, X_{s_p})]$. If the processes X and S are independent, then the conditional expectation $\mathbb{E}[\, f(X_{S_{t_1}}, \ldots, X_{S_{t_p}}) \,|\, \sigma(S_{t_1}, \ldots, S_{t_p})\,]$ is equal to $g(S_{t_1}, \ldots, S_{t_p})$ almost surely. It follows that

$$\mathbb{E}\big[f\big(X_{S_{t_1}}, \ldots, X_{S_{t_p}}\big)\big] \;=\; \mathbb{E}\big[\mathbb{E}\big[f\big(X_{S_{t_1}}, \ldots, X_{S_{t_p}}\big) \,\big|\, \sigma(S_{t_1}, \ldots, S_{t_p})\big]\big] \;=\; \mathbb{E}\big[g(S_{t_1}, \ldots, S_{t_p})\big]\,.$$

The right hand side is determined by the law of X (which determined the function g) and the law of S. Therefore if $\tilde{X}$ and $\tilde{S}$ are two independent processes such that $\tilde{X} \overset{(d)}{=} X$ and $\tilde{S} \overset{(d)}{=} S$, then we have $\mathbb{E}[f(X_{S_{t_1}}, \ldots, X_{S_{t_p}})] = \mathbb{E}[f(\tilde{X}_{\tilde{S}_{t_1}}, \ldots, \tilde{X}_{\tilde{S}_{t_p}})]$. As f is any bounded measurable function, this proves that $(X_{S_{t_1}}, \ldots, X_{S_{t_p}}) \overset{(d)}{=} (\tilde{X}_{\tilde{S}_{t_1}}, \ldots, \tilde{X}_{\tilde{S}_{t_p}})$ for all times $t_1, \ldots, t_p$, and therefore $X\lfloor_S \overset{(d)}{=} \tilde{X}\lfloor_{\tilde{S}}$. The other equality in law $X^{(T)} \overset{(d)}{=} \tilde{X}^{(\tilde{T})}$ can be proved similarly.

(2) The basic idea for this question is that, if $t \mapsto X_t$ is almost surely càdlàg, then it is determined by its restriction on the countable set $\mathbb{Q} \cap [0, \infty)$, and X_T at a continuous random time T can be approximated by X_{T_n}, where T_n is a discretization of T. (The same idea has been used in the proof of the strong Markov property of Lévy processes in the lectures.) More precisely:

For $n \geq 1$, let $T_n = \frac{\lceil nT \rceil}{n}$, where $\lceil x \rceil$ denotes the smallest integer greater or equal to x. It is not hard to see that $T_n \geq T$ and $T_n \to T$ as $n \to \infty$. Since X has almost surely right-continuous paths, we have $X_T = \lim_{n\to\infty} X_{T_n}$ almost surely, and in general for any $t_1, \dots, t_p \geq 0$, $(X_{T+t_1} - X_T, \dots, X_{T+t_p} - X_T) = \lim_{n\to\infty}(X_{T_n+t_1} - X_{T_n}, \dots, X_{T_n+t_p} - X_{T_n})$ almost surely. But, for each n, the random time T_n only takes values in the countable set $\{1/n, 2/n, 3/n, \cdots\}$. Therefore

$$X_{T_n+t_j} = \sum_{k=0}^{\infty} X_{k/n+t_j} \mathbb{1}_{\{T_n = k/n\}}$$

is measurable in $\sigma(X, T)$ for any fixed $t_j \geq 0$. It follows that the vectors $(X_{T_n+t_1} - X_{T_n}, \dots, X_{T_n+t_p} - X_{T_n})$, and therefore $(X_{T+t_1} - X_T, \dots, X_{T+t_p} - X_T)$ are measurable in $\sigma(X, T)$. This shows that the process $X^{(T)}$ is measurable in $\sigma(X, T)$. Similarly, one can show that $(X_{S_{t_1}}, \dots, X_{S_{t_p}})$, and therefore the process $X\lfloor_S$, are measurable in $\sigma(X, S)$.

Exercise 2 (Characterization of Lévy process)**.** Show that a process $X = (X_t)_{t\geq 0}$ is a Lévy process if and only if

(a) for all $\tau \geq 0$, the shifted process $X^{(\tau)}$ is independent of X_τ and has the same law as X, and

(b) there exists a measurable set A of probability 1, such that for all $\omega \in A$, the function $t \mapsto X_t(\omega)$ is càdlàg on $\mathbb{R}_{\geq 0}$.

Solution of Exercise 2. If X is a Lévy process, then it satisfies the conditions (a) (weak Markov property) and (b) (part of the definition of Lévy processes). On the other hand, the condition (a) clearly implies that X has stationary increments, that is, $X_s^{(\tau)} = X_{\tau+s} - X_\tau \overset{(d)}{=} X_s$ for all $\tau, s \geq 0$. It remains to show that (a) also implies that X has independent increments. This can be done by induction:

Let $p \geq 1$ and $0 < t_1 < \cdots < t_p$. We want to deduce from (a) that the random variables $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_p} - X_{t_{p-1}})$ are independent. When $p = 1$, there is nothing to prove. For $p \geq 2$, consider the random vector

$$\mathbf{V} := (X^{(t_1)}_{t_2-t_1}, X^{(t_1)}_{t_3-t_1} - X^{(t_1)}_{t_2-t_1}, \dots, X^{(t_1)}_{t_p-t_1} - X^{(t_1)}_{t_{p-1}-t_1}) = (X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_p} - X_{t_{p-1}})\,.$$

According to (a), $\mathbf{V}$ is independent of X_{t_1}. Moreover, it has the same law as $(X_{t_2-t_1}, X_{t_3-t_1} - X_{t_2-t_1}, \dots, X_{t_p-t_1} - X_{t_{p-1}-t_1})$, whose components are mutually independent by the induction hypothesis. It follows that the p random variables $(X_{t_1}; \mathbf{V}) = (X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_p} - X_{t_{p-1}})$ are mutually independent. This concludes the proof by induction.

Exercise 3 (Lévy process indexed by an independent subordinator)**.** Show that if X is a Lévy process and S a subordinator independent of X, then $X\lfloor_S$ is also a Lévy process. If in addition X is α-stable and S is β-stable, show that $X\lfloor_S$ is $\alpha\beta$-stable.

Solution of Exercise 3. To prove that $X\lfloor_S$ is a Lévy process, let us verify that it satisfies the conditions (a) and (b) of Exercise 2:

(a) Fix $\tau \geq 0$. We want to show that the shifted process $(X\lfloor_S)^{(\tau)}$ is independent of $(X\lfloor_S)_\tau$ and has the same law as $X\lfloor_S$. Observe that $(X\lfloor_S)^{(\tau)} = (X_{S_{\tau+t}} - X_{S_\tau})_{t\geq 0} = (X_{S_{\tau+t}} - X_{S_\tau})_{t\geq 0} = X^{(S_\tau)}\lfloor_{S^{(\tau)}}$, and $(X\lfloor_S)_\tau = X_{S_\tau}$.

Let us start by proving that the triplet $(X^{(S_\tau)}, S^{(\tau)}, X_{S_\tau})$ is independent: consider $\mathbb{E}[f(X^{(S_\tau)})g(S^{(\tau)})h(X_{S_\tau})]$ for bounded measurable functions $f, g : \mathbb{R}^{[0,\infty)} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$. Thanks to the Markov property of S, the random variable S_τ is independent of $S^{(\tau)}$. It is also independent of X. It follows that

$$\mathbb{E}[f(X^{(S_\tau)})g(S^{(\tau)})h(X_{S_\tau})|S_\tau] = F(S_\tau) \qquad \text{where} \qquad F(s) := \mathbb{E}[f(X^{(s)})g(S^{(\tau)})h(X_s)]\,.$$

Since X is a Lévy process independent of S, the triplet $(X^{(s)}, X_s, S^{(\tau)})$ is independent for each $s \geq 0$. Hence we have $F(s) = \mathbb{E}[f(X^{(s)})] \cdot \mathbb{E}[g(S^{(\tau)})] \cdot \mathbb{E}[h(X_s)] = \mathbb{E}[f(X)] \cdot \mathbb{E}[g(S)] \cdot \mathbb{E}[h(X_s)]$ for all $s \geq 0$, where we used the Markov property of both X and S. It follows that

$$\mathbb{E}[f(X^{(S_\tau)})g(S^{(\tau)})h(X_{S_\tau})] = \mathbb{E}[F(S_\tau)] = \mathbb{E}[f(X)] \cdot \mathbb{E}[g(S)] \cdot \mathbb{E}[h(X_{S_\tau})]\,,$$

that is, $X^{(S_\tau)}$, $S^{(\tau)}$ and X_{S_τ} are mutually independent, and $X^{(S_\tau)} \overset{(d)}{=} X$ and $S^{(\tau)} \overset{(d)}{=} S$.

It is not hard to see that the process $X^{(S_\tau)}$ also has almost surely càdlàg paths. Therefore, thanks to Exercise 1(b), $X^{(S_\tau)}\lfloor_{S^{(\tau)}}$ is a measurable function of $X^{(S_\tau)}$ and $S^{(\tau)}$. It follows that $(X\lfloor_S)^{(\tau)} = X^{(S_\tau)}\lfloor_{S^{(\tau)}}$ is independent of X_{S_τ} and has the same distribution as $X\lfloor_S$.

(b) Since X is a Lévy process and S a subordinator, there exists a measurable set A of probability 1, such that for all $\omega \in A$, both $s \mapsto X_s(\omega)$ and $t \mapsto S_t(\omega)$ are càdlàg on $\mathbb{R}_{\geq 0}$, and $t \mapsto S_t(\omega)$ is non-decreasing. On this event A, we have

$$\lim_{\tau \to 0^+} X_{S_{t+\tau}}(\omega) = \lim_{s \to 0^+} X_{S_t+s}(\omega) = X_{S_t}(\omega) \qquad \text{and} \qquad \lim_{\tau \to 0^+} X_{S_{t-\tau}}(\omega) = \lim_{s \to 0^+} X_{S_{t^-}-s}(\omega) = X_{(S_{t^-})^-}(\omega),$$

that is, $t \mapsto X_{S_t}(\omega)$ is càdlàg for all $\omega \in A$. Combining this with (a), we see that $X\lfloor_S$ is a Lévy process.

By definition, a Lévy process X is α-stable if and only if the rescaled process $(\lambda^{-1} X_{\lambda^\alpha s})_{s \geq 0}$ has the same law as X for all $\lambda > 0$. Now if X is α-stable and S is β-stable, then we have $X \overset{(d)}{=} \tilde{X} := (\lambda^{-1} X_{\lambda^\alpha s})_{s\geq 0}$ and $S \overset{(d)}{=} \tilde{S} := (\Lambda^{-1} S_{\Lambda^\beta t})_{t \geq 0}$ for all $\lambda, \Lambda > 0$. It is clear that $\tilde{X}$ and $\tilde{S}$ are independent. Therefore by Exercise 1, $X\lfloor_S \overset{(d)}{=} \tilde{X}\lfloor_{\tilde{S}}$. When $\Lambda = \lambda^\alpha$, the right hand side is equal to $\tilde{X}\lfloor_{\tilde{S}} = (\lambda^{-1} X_{S_{\lambda^{\alpha\beta} t}})_{t \geq 0}$. Since this is true for all $\lambda > 0$, the process $X\lfloor_S$ is $\alpha\beta$-stable.

Exercise 4 (Ladder time process)**.** Let X be a Lévy process with no positive jumps such that $\limsup_{t \to \infty} X_t = \infty$ almost surely. For all $a \geq 0$, let $T_a = \inf\{t \geq 0 \,|\, X_t > a\}$.

(1) Show that almost surely, $T_a < \infty$ and $X_{T_a} = a$ for all $a \geq 0$. (Mind the order of the quantifiers.)

(2) Show that $T = (T_a)_{a \geq 0}$ is a subordinator. (The process T is called the (ascending) ladder time process of X.)

(3) Show that if X is α-stable, then T is $\frac{1}{\alpha}$-stable. What values can α take?

<u>*Solution of Exercise 4.*</u>

(1) The condition $\limsup_{t \to \infty} X_t = \infty$ implies that for all $a \geq 0$, there exists $t \geq 0$ such that $X_t \geq 0$. That is, $T_a < \infty$. Moreover, if $t \mapsto X_t$ is càdlàg on $\mathbb{R}_{\geq 0}$, then the definition of T_a implies that

$$X_{T_a} = \lim_{t \to 0^+} X_{T_a + t} \geq a \qquad \text{and} \qquad X_{T_a^-} = \lim_{t \to 0^+} X_{T_a - t} \leq a\,.$$

Since X has no positive jumps, we also have $X_{T_a} \leq X_{T_a^-}$. If follows that $X_{T_a} = a$.

The above discussion shows that we have $T_a < \infty$ and $X_{T_a} = a$ whenever $\limsup_{t \to \infty} X_t = \infty$ and $t \mapsto X_t$ is càdlàg on $\mathbb{R}_{\geq 0}$. By assumption, the latter conditions are true on a measurable set of probability 1. Therefore almost surely, we have $T_a < \infty$ and $X_{T_a} = a$ for all a.

(2) It is clear that the process T has non-decreasing paths. First, we notice that T_a is a stopping time: Indeed, it is clear that $\{T_a \leq t\} \in \mathcal{F}_{t+\epsilon}$ for all $t \geq 0$ and $\epsilon > 0$, where $(\mathcal{F}_t)_{t \geq 0}$ is the canonical filtration of the Lévy process X, completed with zero probability sets. It follows that $\{T_a \leq t\} \in \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$, where the second equality comes from the right-continuity of the Lévy filtration (which generalizes Blumenthal's zero-one law from $t = 0$ to arbitrary time t). Hence T_a is a stopping time.

Let us show that the process $T = (T_a)_{a \geq 0}$ is a Lévy process using the characterization in Exercise 2:

(a) When the process X has almost surely càdlàg path, we have $T = (\inf\{t \geq 0 : \, X_t > a\})_{a \geq 0} = (\inf\{t \in \mathbb{Q}_{\geq 0} : X_t > a\})_{a \geq 0}$ almost surely. The process on the right hand side depends only on the value of X at countably many times, therefore is a measurable function of the process X. Denote it by $F(X)$. Then we have $T = F(X)$ almost surely for any Lévy process X.

Fix $b \geq 0$. Observe that the shifted process $T^{(b)}$ satisfies

$$T_a^{(b)} \, = \, T_{a+b} - T_b \, = \, \inf\{t \geq 0 : \, X_{T_b + t} > a + b\} \, = \, \inf\{t \geq 0 : \, X_{T_b+t} - X_{T_b} > a\}\ ,$$

that is, $T_a^{(b)} = \inf\{t \geq 0 : \, X_t^{(T_b)} > a\}$ for all $a \geq 0$. Since $X^{(T_b)}$ is a Lévy process, we have $T^{(b)} = F(X^{(T_b)})$ almost surely. By the strong Markov property, we have $X^{(T_b)} \overset{(d)}{=} X$. Applying the mapping F to both sides gives $T^{(b)} \overset{(d)}{=} T$. Moreover, $X^{(T_b)}$ is independent of the σ-field $\mathcal{F}_{T_b}$. But T_b is $\mathcal{F}_{T_b}$-measurable. Hence the process $X^{(T_b)}$ is independent of T_b, and therefore $T^{(b)} = F(X^{(T_b)})$ is also independent of T_b.

Notice that the identity in distribution $T_0 \overset{(d)}{=} T_0^{(b)} = T_b - T_b$ guarantees that $T_0 = 0$ almost surely.

(b) Since the function $a \mapsto T_a$ is non-decreasing, it has left and right limit at every point. On the other hand, the definition of T_a implies that for all $\epsilon > 0$, there exists $t \in (T_a, T_a + \epsilon)$ such that $X_t > a$. For any $b \in (a, X_t)$, we have $T_a \leq T_b \leq t < T_a + \epsilon$. This shows that $\lim_{b \to a^+} T_b = T_a$. Hence the paths of T are almost surely càdlàg.

(3) Let $\tilde{X} = (\lambda^{-1} X_{\lambda^\alpha t})_{t\geq 0}$. The ladder time process of $\tilde{X}$ is given by $\tilde{T}_a = \inf\{t \geq 0 : X_{\lambda^\alpha t} > \lambda a\} = \lambda^{-\alpha} T_{\lambda a}$. According to the discussion in Part (a) of Question (2), we have $\tilde{T} = F(\tilde{X})$ almost surely. Now if $\tilde{X}$ is α-stable, then $X \overset{(d)}{=} \tilde{X}$ for all $\lambda > 0$. Applying F to boths sides gives that $T \overset{(d)}{=} \tilde{T}$ for all $\lambda > 0$, which implies that T is $\frac{1}{\alpha}$-stable.

Since T is a $\frac{1}{\alpha}$-stable subordinator, we must have $\frac{1}{\alpha} \in (0, 1)$, therefore $\alpha \in (1, 2]$.

Indeed, X cannot be an α-stable Lévy process with $\alpha \in (0, 1]$, since in that case having no positive jumps would imply that $-X$ is a subordinator (c.f. the construction of general α-stable processes in the lectures, see also Exercise 5 and 6), which contradicts the assumption that $\limsup_{t\to\infty} X_t = \infty$ almost surely. On the other hand, for $\alpha \in (1, 2]$, the α-stable Lévy processes are martingales and we always have $\limsup_{t\to\infty} X_t = \infty$ almost surely. (Interested readers may try to prove the following fact: for the general Lévy process X, either $\limsup_{t\to\infty} X_t = \infty$ almost surely, or $\lim_{t\to\infty} X_t = -\infty$ almost surely.)

Exercise 5 (Cauchy process). Recall that the Cauchy process C is the Lévy process with characteristic exponent $\Psi(\lambda) = -|\lambda|$.

(1) Show that the marginal distribution of the Cauchy process is given by $\mathbb{P}(C_t \in [x, x + \mathrm{d}x]) = \frac{t}{t^2+x^2} \frac{\mathrm{d}x}{\pi}$.
(One can use the Fourier transform formula that $\int_{\mathbb{R}} \frac{e^{i\lambda x}}{1+x^2} \mathrm{d}x = \pi e^{-|\lambda|}$ for all $\lambda \in \mathbb{R}$).

(2) What is the Lévy-Khinchine triplet (a, b, Π) of the Cauchy process?

(3) Show that a Lévy process X is 1-stable if and only if it is of the form $X_t = u \cdot C_t + bt$, where $u \geq 0$ and $b \in \mathbb{R}$.
(Notice that the Lévy measure of X has to be symmetric. Compare this to the α-stable processes with $\alpha \in (0, 1) \cup (1, 2)$.)

(4) Let $B = (B_s)_{s\geq 0}$ be the standard Brownian motion, and T be a $\frac{1}{2}$-stable subordinator independent of B. Show that $B\lfloor_T$ is a multiple of the Cauchy process.
(Remark: compare this to the case where T_a is defined as the hitting time of level a by the Brownian motion B.)

Solution of Exercise 5.

(1) By the definition of the characteristic exponent, we have $\mathbb{E}[e^{i\lambda C_t}] = e^{t\Psi(\lambda)} = e^{-|t\lambda|}$ for all $t \geq 0$. Using the Fourier transform formula provided in the question, we see that

$$e^{-|t\lambda|} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i\lambda t x}}{1 + x^2} \mathrm{d}x = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i\lambda x}}{1 + (x/t)^2} \frac{\mathrm{d}x}{t} = \int_{\mathbb{R}} e^{i\lambda x} \frac{t}{t^2 + x^2} \frac{\mathrm{d}x}{\pi}$$

for all $\lambda \in \mathbb{R}$. Since the characteristic function of a random variable uniquely determines its distribution, we have $\mathbb{P}(C_t \in [x, x + \mathrm{d}x]) = \frac{t}{t^2+x^2} \frac{\mathrm{d}x}{\pi}$.

(2) We have seen briefly in the lectures that the Lévy measure of an α-stable Lévy process must be of the form $\Pi(\mathrm{d}x) = (c_+ \mathbb{1}_{\{x>0\}} + c_- \mathbb{1}_{\{x<0\}}) \frac{\mathrm{d}x}{|x|^{\alpha+1}}$ for some $c_+, c_- \geq 0$. Let us briefly rederive this fact here:

By the Lévy-Khintchine characterization theorem, all Lévy process can be written as the sum of three mutually independent processes $aB + bt + J$, where $a, b \in \mathbb{R}$, B is the standard Browian motion, and $J = X^{<1} + X^{\geq 1}$ is a jump process with intensity Π plus compensation for the small jumps. According to its construction, the jump process J makes almost surely a finite number of positive jumps of size $\geq x$ before time t, and the number of such jumps is a Poisson random variable of mean $t \cdot \Pi([x, \infty))$, for any $x, t > 0$. Similarly, the number of negative jumps of size $\leq -x$ before time t is a Poisson random variable of mean $t \cdot \Pi((-\infty, -x])$. Since the jumps of J are exactly those of X, we can read the Lévy measure of a Lévy process X from the statistics of its jumps.

When X is an α-stable Lévy process, we have for all $\lambda > 0$

$$\Pi([\lambda, \infty)) = \mathbb{E}[\text{number of jumps of } X \text{ of size} \geq \lambda \text{ before time } 1] .$$

But since X has the same law as $(\lambda X_{\lambda^{-\alpha} t})_{t\geq 0}$, the right hand side is also equal to

$$\Pi([\lambda, \infty)) = \mathbb{E}[\text{number of jumps of } X \text{ of size} \geq 1 \text{ before time } \lambda^{-\alpha}] .$$

It follows that $\Pi([\lambda, \infty)) = \lambda^{-\alpha} \cdot \Pi([1, \infty))$ for all $\lambda > 0$. Similarly, we have $\Pi((-\infty, -\lambda]) = \lambda^{-\alpha} \cdot \Pi((-\infty, -1])$ for all $\lambda > 0$. These two properties imply that Π must be of the form $\Pi(\mathrm{d}x) = (c_+ \mathbb{1}_{\{x>0\}} + c_- \mathbb{1}_{\{x<0\}}) \frac{\mathrm{d}x}{|x|^{\alpha+1}}$ for some $c_+, c_- \geq 0$.

Now let us get back to the Cauchy process: since its characteristic exponent $\Psi(\lambda)$ is proportional to $|\lambda|$, the Cauchy process is a 1-stable Lévy process. Moreover, Question 1 shows that the law of the Cauchy process is symmetric with respect to 0. Therefore its Lévy measure must be of the form $\Pi = c \cdot \frac{\mathrm{d}x}{x^2}$ for some $c > 0$. Plugging this into the Lévy-Khintchine formula and combining the integrals on $x > 0$ with those on $x < 0$, we obtain that for all $\lambda > 0$:

$$\Psi(\lambda) \;=\; -\frac{a^2}{2}\lambda^2 + ib\lambda + c \cdot \int_0^\infty \big(e^{i\lambda x} + e^{-i\lambda x} - 2\big)\,\frac{\mathrm{d}x}{x^2} \;=\; -\frac{a^2}{2}\lambda^2 + ib\lambda + 2c\cdot\lambda \int_0^\infty (\cos y - 1)\frac{\mathrm{d}y}{y^2}\,.$$

Since $\Psi(\lambda) = -\lambda$ for all $\lambda > 0$, we have $a = b = 0$ and $c = \big(2\int_0^\infty (1-\cos y)\mathrm{d}y/y^2\big)^{-1}$. Thanks to the formula $\int_0^\infty \frac{\sin^2(u)}{u^2}\mathrm{d}u = \frac{\pi}{2}$, we have $c = \pi^{-1}$. Thus the Lévy-Khintchine triplet of the Cauchy process is $(a, b, \Pi) = (0, 0, \frac{1}{\pi}\frac{\mathrm{d}x}{x^2})$.

(3) As shown in the previous question, if X is a 1-stable Lévy process, then its Lévy measure must be of the form $\Pi(\mathrm{d}x) = (c_+\mathbb{1}_{\{x>0\}} + c_-\mathbb{1}_{\{x<0\}})\frac{\mathrm{d}x}{x^2}$, and for all $\lambda > 0$, its characteristic exponent of the form

$$\begin{aligned}
\Psi(\lambda) &= ib\lambda + c_+\int_0^\infty \big(e^{i\lambda x} - 1 - i\lambda x\mathbb{1}_{\{x<1\}}\big)\frac{\mathrm{d}x}{x^2} + c_-\int_0^\infty \big(e^{-i\lambda x} - 1 + i\lambda x\mathbb{1}_{\{x<1\}}\big)\frac{\mathrm{d}x}{x^2}\\
&= \left(ib + c_+\int_0^\infty \big(e^{iy} - 1 - iy\mathbb{1}_{\{y<\lambda\}}\big)\frac{\mathrm{d}y}{y^2} + c_-\int_0^\infty \big(e^{-iy} - 1 + iy\mathbb{1}_{\{y<\lambda\}}\big)\frac{\mathrm{d}y}{y^2}\right)\cdot\lambda\\
&= \left(ib + c_+\int_0^\infty \big(e^{iy} - 1 - iy\mathbb{1}_{\{y<1\}}\big)\frac{\mathrm{d}y}{y^2} + c_-\int_0^\infty \big(e^{-iy} - 1 + iy\mathbb{1}_{\{y<1\}}\big)\frac{\mathrm{d}y}{y^2} - (c_+ - c_-)\int_1^\lambda iy\frac{\mathrm{d}y}{y^2}\right)\cdot\lambda\\
&= \big(\Psi(1) - (c_+ - c_-)\log\lambda\big)\cdot\lambda
\end{aligned}$$

But since X is 1-stable, we must have $\Psi(\lambda) = \lambda\Psi(1)$ for all $\lambda > 0$. It follows that $c_+ = c_-$. So the Lévy-Khintchine triplet of X is of the form $(0, b, \frac{u}{\pi}\frac{\mathrm{d}x}{x^2})$ for some $b \in \mathbb{R}$ and $u \geq 0$, which corresponds to $X_t = u \cdot C_t + bt$.

(4) Since B is a 2-stable Lévy process, and the subordinator T is $1/2$-stable and independent of B, according to Exercise 3, $B\lfloor_T$ is a 1-stable Lévy process. Moreover, since the law of B is symmetric with respect to 0, so is the law of $B\lfloor_T$. It follows that the 1-stable process does not have a drift component, and thus is of the form $u \cdot C$.

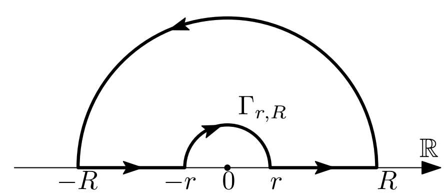
Remark that the independence between B and T is crucial for the conclusion to hold. Indeed, by Exercise 4, the ladder time process $S_a = \inf\{s \geq 0 :\ B_s > a\}$ of the Brownian motion B is also a $\frac{1}{2}$-stable subordinator, but in this case $B\lfloor_S$ is the drift process $B_{S_a} = a$.

Exercise 6 (Charateristic exponent of stable processes). Let $\alpha \in (0,1)\cup(1,2)$ and $c_+, c_- \geq 0$. Let X be the α-stable process of Lévy measure $\Pi(\mathrm{d}x) = (c_+\mathbb{1}_{\{x>0\}} + c_-\mathbb{1}_{\{x<0\}})\frac{\mathrm{d}x}{|x|^{\alpha+1}}$.

(1) Show that the characteristic exponent Ψ of X can be written as

$$\Psi(\lambda) = \begin{cases} |\lambda|^\alpha\cdot\big(c_+\Psi_+(1) + c_-\Psi_+(-1)\big) & \text{if } \lambda \geq 0\\ |\lambda|^\alpha\cdot\big(c_+\Psi_+(-1) + c_-\Psi_+(1)\big) & \text{if } \lambda \leq 0,\end{cases}$$

where Ψ_+ is the characteristic exponent of the α-stable process of Lévy measure $\mathbb{1}_{\{x>0\}}\frac{\mathrm{d}x}{x^{\alpha+1}}$.



(2) Let $\Gamma_{r,R}$ be the oriented closed contour in the upper half-plane defined in the picture above.
When $\alpha \in (0,1)$, show that the integral of $z \mapsto \frac{e^{iz}-1}{z^{\alpha+1}}$ along the contour $\Gamma_{r,R}$ has a finite limit as $r \to 0$ and $R \to \infty$.

(3) Deduce that $\Psi_+(-1) = -e^{i\pi\alpha}\Psi_+(1)$ when $\alpha \in (0,1)$. Show that the same relation also holds when $\alpha \in (1,2)$.

(4) Deduce that the characteristic exponent of X can be written as $\Psi(\lambda) = -c|\lambda|^\alpha\cdot\big(1 - \mathrm{sgn}(\lambda)\cdot\beta\tan\big(\frac{\pi\alpha}{2}\big)\big)$, where $\beta = \frac{c_+-c_-}{c_++c_-}$ and $c > 0$ is a constnat that only depends on c_+, c_- and α.

Solution of Exercise 6.

(1) We have seen in the lectures that when $\alpha \in (0,1)\cap(1,2)$, there is indeed (in distribution) a unique α-stable process X_+ of Lévy measure $\mathbb{1}_{\{x>0\}}\frac{\mathrm{d}x}{x^{\alpha+1}}$. Its characteristic exponent satisfies $\Psi_+(\lambda) = |\lambda|^\alpha\cdot\Psi_+(\mathrm{sgn}(\lambda))$ for all $\lambda \in \mathbb{R}$. Thanks to the Lévy-Khintchine formula, the relation between the Lévy measures of X and X_+ implies that

$$\Psi(\lambda) \;=\; c_+\Psi_+(\lambda) + c_-\Psi_+(-\lambda) \;=\; |\lambda|^\alpha\cdot\big(c_+\Psi_+(\mathrm{sgn}(\lambda)) + c_-\Psi_+(-\mathrm{sgn}(\lambda))\big)$$

for all $\lambda \in \mathbb{R}$. This is the same as formula to be proven.

(2) When $\alpha \in (0,1)$, the function $\frac{e^{iz}-1}{z^{\alpha+1}}$ is bounded by $\frac{2}{|z|^{\alpha+1}}$ uniformly when $|z| \to \infty$ in the upper half-plane, and bounded by $\frac{2}{|z|^{\alpha}}$ uniformly when $z \to 0$. It follows that the integral of $z \mapsto \frac{e^{iz}-1}{z^{\alpha+1}}$ on the two circular parts of the contour $\Gamma_{r,R}$ converges to zero when $r \to 0$ and $R \to \infty$. Therefore

$$\oint_{\Gamma_{r,R}} \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z \xrightarrow[r\to 0, R\to\infty]{} \int_0^\infty \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z + \int_{-\infty}^0 \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z$$

(3) Since the function $z \mapsto \frac{e^{iz}-1}{z^{\alpha+1}}$ is holomorphic in the domain inside the contour $\Gamma_{r,R}$. Thus we have by Cauchy's integral theorem $\oint_{\Gamma_{r,R}} \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z = 0$ for all $R > r > 0$. It follows that

$$\int_0^\infty \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z + \int_{-\infty}^0 \frac{e^{iz}-1}{z^{\alpha+1}} \mathrm{d}z = 0\,. \tag{1}$$

In the analytic branch of $z \mapsto \frac{e^{iz}-1}{z^{\alpha+1}}$ in the upper half-plane, we have $\frac{e^{i(-x)}-1}{(-x)^{\alpha+1}} = e^{-i\pi(\alpha+1)} \frac{e^{-ix}-1}{x^{\alpha+1}}$ for all $x > 0$. Therefore the second integral in (1) simplifies to $e^{-i\pi\alpha} \int_0^\infty \frac{e^{-ix}-1}{x^{\alpha+1}} \mathrm{d}x$. We recognize that $\int_0^\infty \frac{e^{ix}-1}{x^{\alpha+1}} \mathrm{d}x = \Psi_+(1)$ and $\int_0^\infty \frac{e^{-ix}-1}{x^{\alpha+1}} \mathrm{d}x = \Psi_+(-1)$. Hence the equation (1) implies $\Psi_+(1) + e^{-i\pi\alpha}\Psi_+(-1) = 0$.

When $\alpha \in (1,2)$, we use the function $z \mapsto \frac{e^{iz}-1-iz}{z^{\alpha+1}}$. One can check that its contour integral on $\Gamma_{r,R}$ also converges when $r \to 0$ and $R \to \infty$. Then the same computation shows that $\Psi(1) + e^{-i\pi\alpha}\Psi_+(-1) = 0$.

(4) We plug the above relation between $\Psi_+(1)$ and $\Psi_+(-1)$ into the expression in Question 1. When $\lambda > 0$, it gives

$$\begin{aligned} \Psi(\lambda) &= |\lambda|^\alpha \cdot e^{i\frac{\pi\alpha}{2}} \Psi_+(1) \cdot \left(c_+ e^{-i\frac{\pi\alpha}{2}} - c_- e^{i\frac{\pi\alpha}{2}}\right) \\ &= |\lambda|^\alpha \cdot e^{i\frac{\pi\alpha}{2}} \Psi_+(1) \cdot \cos(\pi\alpha/2) \cdot \left((c_+ + c_-) - (c_+ - c_-)\tan(\pi\alpha/2)\right) \\ &= |\lambda|^\alpha \cdot e^{i\frac{\pi\alpha}{2}} \Psi_+(1) \cdot (c_+ + c_-)\cos(\pi\alpha/2) \cdot \left(1 - \beta\tan(\pi\alpha/2)\right) \end{aligned}$$

where $\beta = \frac{c_+ - c_-}{c_+ + c_-}$. When $\lambda < 0$, one can check that only the sign in front of β changes. Therefore we have $\Psi(\lambda) = c \cdot |\lambda|^\alpha \cdot \left(1 - \mathrm{sgn}(\lambda) \cdot \beta \tan\left(\frac{\pi\alpha}{2}\right)\right)$ for all $\lambda \in \mathbb{R}$, where $c = e^{i\frac{\pi\alpha}{2}} \Psi_+(1) \cdot (c_+ + c_-)\cos(\pi\alpha/2)$ only depends on α and c_+, c_-.

Exercise 7. Let $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ be the upper half-plane, and $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$ its closure. Show that the characteristic exponent of a subordinator can be extended to a continuous function $\Psi : \overline{\mathbb{H}} \to \mathbb{C}$ that is analytic in $\mathbb{H}$.

Solution of Exercise 7. The characteristic exponent Ψ of a subordinator X satisfies $e^{\Psi(\lambda)} = \mathbb{E}[e^{i\lambda X_1}] = \int_0^\infty e^{i\lambda x}\mu_1(\mathrm{d}x)$, where μ_1 denotes the law of X_1. The distribution μ_1 is supported on $[0,\infty)$ since X is a subordiantor. For $\lambda \in \mathbb{C}$ and $x \geq 0$, we have $|e^{i\lambda x}| = e^{-\mathrm{Im}(\lambda)x}$. Therefore

- For all $\lambda \in \overline{\mathbb{H}}$, the function $x \mapsto |e^{i\lambda x}|$ is bounded by 1. Hence $\lambda \mapsto \int_0^\infty e^{i\lambda x}\mu_1(\mathrm{d}x)$ is continuous on $\overline{\mathbb{H}}$.
- Fix any $\epsilon > 0$. For all $\lambda \in \mathbb{H}_\epsilon := \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > \epsilon\}$, the function $x \mapsto \left|\frac{\mathrm{d}}{\mathrm{d}\lambda} e^{i\lambda x}\right|$ is bounded by $xe^{-\epsilon x}$, which is integrable with respect to $\mu_1(\mathrm{d}x)$. Hence $\lambda \mapsto \int_0^\infty e^{i\lambda x}\mu_1(\mathrm{d}x)$ has a well-defined complex derivative when $\lambda \in \mathbb{H}_\epsilon$.

It follows that $e^{\Psi(\lambda)}$ extends to a continuous function on $\overline{\mathbb{H}}$ that is analytic on $\mathbb{H}$. Using the argument seen in the lectures, that is, $\mathbb{E}[e^{i\lambda X_1}] = \mathbb{E}[e^{i\lambda X_{1/n}}]^n$ for all $n \geq 1$ and $\mathbb{E}[e^{i\lambda X_{1/n}}] \to \mathbb{E}[e^{i\lambda X_0}] = 1$ as $n \to \infty$, we see that $\mathbb{E}[e^{i\lambda X_1}] \neq 0$ for all $\lambda \in \overline{\mathbb{H}}$. It follows that its logarithm Ψ is well-defined, continuous on $\overline{\mathbb{H}}$, and analytic on $\mathbb{H}$.

Exercise 8 (Gamma process)**.** For $s \in \mathbb{Z}_{\geq 1}$ and $t > 0$, let μ_s^t be the law of $\|B_t\|^2$, where $B = (B^1, \cdots, B^s)$ is the standard Brownian motion in $\mathbb{R}^s$, and $\|\cdot\|$ is the L^2 norm of $\mathbb{R}^s$. (The process $\|B\|^2$ is called the *squared Bessel process* of dimension s.)

(1) Compute the density function of μ_s^t. (Hint: the definition of the Gamma function for $a > 0$ is $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\mathrm{d}x$.)

Notice that the above density function makes sense for all $s > 0$. From now on, we use it to define μ_s^t for all $s \in (0,\infty)$.

(2) Show that if X and Y are independent random variables of laws μ_s^t, $\mu_{s'}^t$ respectively, then $X + Y$ has the law $\mu_{s+s'}^t$.

(3) Deduce that μ_s^t is infinitely divisible for all $s > 0$ and $t > 0$.

Let $\gamma^t = (\gamma^t_s)_{s\geq 0}$ be a Lévy process such that μ^t_s is the law of γ^t_s for all $s > 0$. (For this we admit the following general result: a probability distribution μ on $\mathbb{R}$ is infinitely divisible if and only if there is a Lévy process X such that μ is the law of X_1. In the case of subordinators, this statement is actually simple to prove, and we leave it as a simple additional exercise!)
The process γ^t is called a *Gamma process*.

(4) Show that γ^t is a subordinator, and compute its characteristic exponent $\Psi^t(\lambda)$.
(Hint: one may carry out the computation for $\lambda \in i\mathbb{R}_{\geq 0}$ and apply the result of Exercise 7.)

(5) Check that the γ^t is the pure jump subordinator of Lévy measure $\Pi^t(\mathrm{d}x) = \frac{1}{2x}e^{-x/2t}\mathrm{d}x$.

Solution of Exercise 8.

(1) The law of the Gaussian vector $(B^1_t, \cdots, B^s_t)$ is given by $\left(\frac{1}{2\pi t}\right)^{s/2}\exp\left(-\frac{x_1^2+\cdots+x_s^2}{2t}\right)\mathrm{d}x_1\cdots\mathrm{d}x_s$. Thanks to the rotation invariance of the Lebesgue measure, it can be written in the polar coordinate as $\left(\frac{1}{2\pi t}\right)^{s/2}\exp\left(-\frac{r^2}{2t}\right)\cdot r^{s-1}\mathrm{d}r\mathrm{d}\sigma$, where σ is the area measure on the unit sphere $\mathbb{S}^{s-1}\subset\mathbb{R}^s$. Hence for any bounded measurable function f, we have

$$\mathbb{E}\left[f(\|B_t\|^2)\right] = \left(\frac{1}{2\pi t}\right)^{s/2}\int_0^\infty \mathrm{d}r\int_{\mathbb{S}^{s-1}}\mathrm{d}\sigma\cdot e^{-r^2/2t}r^{s-1}\cdot f(r^2)\,.$$

We integrate out the angular variables and make the change of variable $r^2 = x$. After simplification, we obtain

$$\mathbb{E}\left[f(\|B_t\|^2)\right] = c^t_s\int_0^\infty \mathrm{d}x\cdot e^{-x/2t}x^{s/2-1}\cdot f(x) \tag{2}$$

for some normalization constant c^t_s. We determine c^t_s by setting $f = 1$ in the above equation, that is

$$c^t_s = \left(\int_0^\infty x^{s/2-1}e^{-x/2t}\mathrm{d}x\right)^{-1} = \left((2t)^{s/2}\int_0^\infty x^{s/2-1}e^{-x}\mathrm{d}x\right)^{-1} = \left((2t)^{s/2}\Gamma(s/2)\right)^{-1}.$$

Then it follows from (2) that $\mu^t_s(\mathrm{d}x) = \rho^t_s(x)\mathrm{d}x$, with the density function

$$\rho^t_s(x) = \frac{1}{(2t)^{s/2}\Gamma(s/2)}x^{s/2-1}e^{-x/2t}\,.$$

(This is known as a *Gamma distribution*.)

(2) When s and s' are positive integers, the result is immediate: indeed, if $\tilde{B}$ is a standard Brownian motion in $\mathbb{R}^{s+s'}$, then its first s components $B = (\tilde{B}^1, \cdots, \tilde{B}^s)$ and last s' components $B' = (\tilde{B}^{s+1}, \cdots, \tilde{B}^{s+s'})$ are two independent standard Brownian motions in $\mathbb{R}^s$ and $\mathbb{R}^{s'}$, respectively. Therefore $\|B_t\|^2$ and $\|B'_t\|^2$ are two independent random variables of laws μ^t_s and $\mu^t_{s'}$ respectively, and $\|\tilde{B}_t\|^2 = \|B_t\|^2 + \|B'_t\|^2$ indeed follows the law $\mu^t_{s+s'}$.

For general values of $s, s' \geq 0$, let ν be the law of $X + Y$ for two independent random variables X, Y of distributions μ^t_s and $\mu^t_{s'}$. For any bounded measurable function f, we have

$$\mathbb{E}[f(X+Y)] = \int_{[0,\infty)^2} f(x+y)\cdot\rho^t_s(x)\rho^t_{s'}(y)\,\mathrm{d}x\mathrm{d}y = \int_0^\infty f(z)\cdot\left(\int_0^z \rho^t_s(x)\cdot\rho^t_{s'}(z-x)\,\mathrm{d}x\right)\mathrm{d}z\,.$$

Therefore the distribution ν has a density given by the convolution

$$\begin{aligned}\int_0^z \rho^t_s(x)\cdot\rho^t_{s'}(z-x)\,\mathrm{d}x &= c^t_{s,s'}\int_0^z x^{s/2-1}(z-x)^{s'/2-1}e^{-z/2t}\mathrm{d}x\\ &= c^t_{s,s'}\left(\int_0^1 u^{s/2-1}(1-u)^{s'/2-1}\mathrm{d}x\right)\cdot z^{(s+s')/2-1}e^{-z/2t}\,,\end{aligned}$$

where $c^t_{s,s'}$ is some constant that does not depend on z. We see that the density function of ν is proportional to that of $\mu^t_{s+s'}$. Since both are probability measures, we must have $\nu = \mu^t_{s+s'}$.

(3) According to the previous question, for all $s, t > 0$ and integer $n \geq 1$, a random variable X of law μ^t_s can be written as the sum of n i.i.d. random variables of law $\mu^t_{s/n}$. Therefore the distribution μ^t_s is infinitely disivible.

(4) Since the distribution μ_s^t is supported on $(0,\infty)$, the Lévy process γ^t is almost surely non-negative at each time, therefore it is a subordinator. By definition, its characteristic exponent Ψ^t satisfies

$$e^{\Psi^t(\lambda)} \;=\; \int_0^\infty e^{i\lambda x}\mu_1^t(\mathrm{d}x) \;=\; \frac{1}{\sqrt{2t}\,\Gamma(1/2)}\int_0^\infty e^{i\lambda x}x^{-1/2}e^{-x/2t}\mathrm{d}x\,. \;=\; \frac{1}{\Gamma(1/2)}\int_0^\infty e^{2i\lambda t\cdot x}x^{-1/2}e^{-x}\mathrm{d}x\,.$$

To evaluate the integral, let us consider a $\lambda = ip$ for $p \geq 0$, as indicated in the hint. Then we have

$$e^{\Psi^t(ip)} \;=\; \frac{1}{\Gamma(1/2)}\int_0^\infty x^{-1/2}e^{-(1+2tp)\cdot x}\mathrm{d}x \;=\; \frac{1}{\sqrt{1+2pt}\,\Gamma(1/2)}\int_0^\infty x^{-1/2}e^{-x}\mathrm{d}x \;=\; \frac{1}{\sqrt{1+2pt}}\,.$$

Therefore $\Psi^t(ip) = -\frac{1}{2}\log(1+2pt)$ for all $p \geq 0$. By Exercise 7, we know that Ψ^t is analytic on the upper half-plane, it follows that $\Psi^t(\lambda) = \frac{1}{2}\log(1-2i\lambda t)$ for all $\lambda \in \overline{\mathbb{H}} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq 0\}$. When $\lambda \in \mathbb{R}$, one can check that this is equivalent to

$$\Psi^t(\lambda) = -\frac{1}{4}\log\left(1+(2\lambda t)^2\right) + \frac{i}{2}\arctan(2\lambda t)\,.$$

(5) According to the Lévy-Khintchine formula, we just need to check that $\Psi^t(\lambda) = \int_0^\infty (e^{i\lambda x}-1)\Pi^t(\mathrm{d}x)$. As in the previous question, let us check the relation when $\lambda = ip$ for some $p \geq 0$: For $p \geq 0$, let

$$I(p) \;:=\; \int_0^\infty (e^{-px}-1)\Pi^t(\mathrm{d}x) \;=\; \int_0^\infty (e^{-px}-1)\cdot e^{-x/2t}\,\frac{\mathrm{d}x}{2x}\,.$$

One can check that $I(p)$ is continously differentiable on $(-\frac{1}{2t},\infty)$, and the derivative with respect to p commutes with the integral over x. It follows that for all $p > -\frac{1}{2t}$, we have

$$I'(p) \;=\; \int_0^\infty (-xe^{-px})\cdot e^{-x/2t}\,\frac{\mathrm{d}x}{2x} \;=\; -\frac{1}{2}\cdot\frac{1}{p+1/2t}\,.$$

Integrating both sides with respect to p while taking into account $I(0)=0$ gives that $I(p) = -\frac{1}{2}\log(1+2pt) = \Psi^t(ip)$ for all $p \geq 0$. By the uniqueness of analytic continuation, this actually proves that $\int_0^\infty (e^{i\lambda x}-1)\Pi^t(\mathrm{d}x) = \Psi^t(\lambda)$ for all λ in the upper half-plane. So we can conclude that γ^t is a pure jump subordinator of Lévy measure Π^t.