The isoperimetric problem in $H^m_{\mathbb{C}}$ via the hyperbolic log-convex density conjecture

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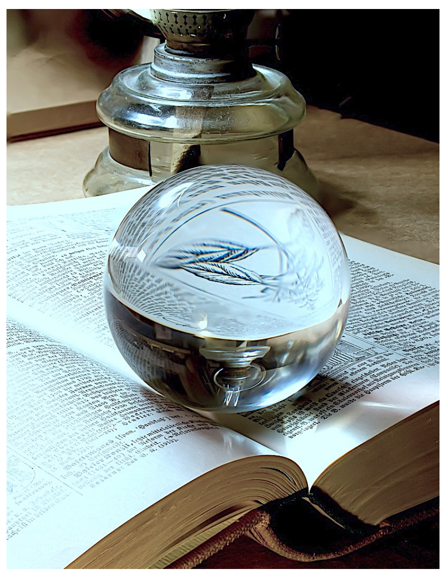
The problem

Let M be a Riemannian manifold.

A set $E \subset M$ solves the isoperimetric problem if it minimizes the perimeter among all sets with equal volume.



Example:
Geodesic balls are uniquely isoperimetric in $\mathbb{R}^n$, S^{n-1}, and $H^n_{\mathbb{R}}$.

Open Conjecture [3]

Gromov-Ros: Geodesic balls are isoperimetric for all volumes in the complex hyperbolic space $H^m_{\mathbb{C}}$.

Hopf-symmetric sets

Def: A set $E \subset H^m_{\mathbb{C}} = \mathrm{SU}(m,1)/\mathrm{U}(m)$ is *Hopf-symmetric* if it is invariant under the action of the one parameter subgroup $e^{i\theta} Id_m \leqslant \mathrm{U}(m)$ up to isometry.

Example: The set E such that

$$\partial E = \{\exp_o(\rho(x)x) : x \in S^{2m-1} \subset T_o H^m_{\mathbb{C}}\}$$

with $\rho \in C^1(S^{2n-1}, \mathbb{R}_+)$ constant along the Hopf fibration

$$h : S^1 \to S^{2m-1} \to P^{m-1}_{\mathbb{C}}.$$

Our contribution [8]

Theorem A: In $H^m_{\mathbb{C}}$, geodesic balls are uniquely isoperimetric in the class of Hopf-symmetric sets for all volumes.

Main observation

Def: $f : H^n_{\mathbb{R}} \to \mathbb{R}$ is *(strictly) radially log-convex* if $\exists\, o \in H^n_{\mathbb{R}}$ such that $\ln(f(x)) = h(d(o,x))$ for $h : \mathbb{R} \to \mathbb{R}$ (strictly) convex and even.

Via a comparison argument between $H^m_{\mathbb{C}}$ and $H^{2m}_{\mathbb{R}}$, Theorem A is implied by

Theorem B: Centered geodesic balls are uniquely isoperimetric in $H^n_{\mathbb{R}}$ with respect to the *weighted* volume and perimeter

$$V_f(E) = \int_E f \, d\mathscr{H}^n, \qquad P_f(E) = \int_{\partial E} f \, d\mathscr{H}^{n-1},$$

if f is strictly radially and log-convex.

Background

Brakke conjectured Theorem B in the Euclidean context, which inspired multiple results, notably [2, 4, 6, 7]. Recently, G. R. Chambers proved Brakke's conjecture in [1]. For a partial contribution in $H^n_{\mathbb{R}}$, see [5].

Proof of Theorem B

We adapt Chamber's strategy.

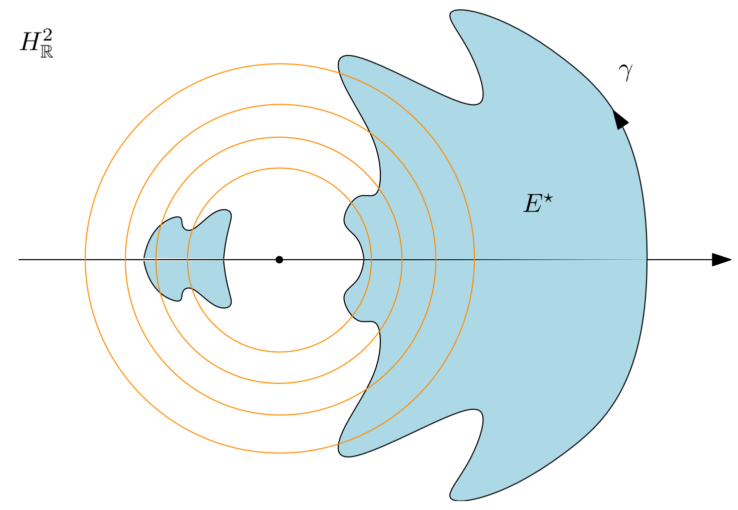
1) There exists an optimal set. It is bounded and C^∞ away from a set of higher codimension.
2) The first variation of P_f: at each regular point of ∂E

$$\mathbf{H}_f := H + \partial_\nu \ln(f) = \lambda, \qquad (1)$$

is constant, where H is the Riemannian mean curvature.
3) f radial: $E^\star$ spherical symmetrization of E satisfies

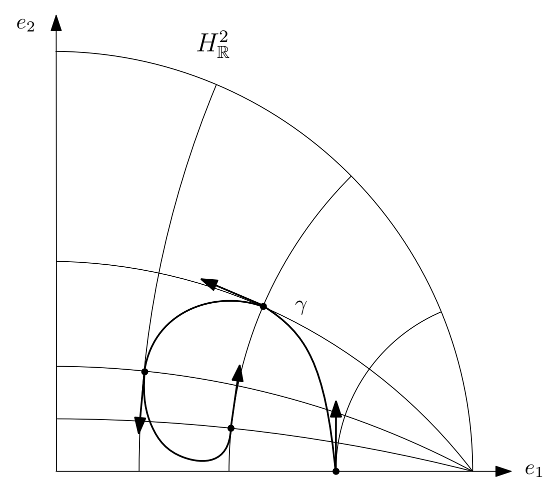
$$V_f(E^\star) = V_f(E), \qquad P_f(E^\star) \leq P_f(E).$$

4) The profile of $E^\star$ is a smooth curve γ in $H^2_{\mathbb{R}}$. By (1) it solves an explicit ODE.



Spherical profile.

5) A meticulous analysis of γ shows that either it is a centered circle (and therefore E is a ball), or it does a curl



The curl.

contradicting the fact that $E^\star$ is spherically symmetric.

Remark: Theorem A holds in all rank one symmetric spaces of non-compact type $H^m_{\mathbb{C}}, H^m_{\mathbb{H}}, H^2_{\mathbb{O}}$.

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