D-BAUG | Analysis I and II | Summer 2013
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1. **a)** Determine all solutions $z \in \mathbb{C}$ of the equation

$$z^4 = -9 + 9\sqrt{3}i$$

and draw them in the complex plane.

b) Determine and draw the region M in the complex plane $\mathbb{C}$

$$M := \{z \in \mathbb{C} \,|\, |z + 1 - 2i| \leq |z - 4 + i| \wedge \mathrm{Re}(z) < 0 \wedge \mathrm{Im}(z) < 0\}\,.$$

2. The movement of a particle is given by the following parametric curve in which $t \in \mathbb{R}$ is to be interpreted as time:

$$\alpha(t) = (3\cosh t, 2\sinh t).$$

a) Determine the cartesian equation of the curve (that is, in terms of x and y) and compute the slope of its asymptotes for $t \to \pm\infty$.

b) What is the minimal velocity of the particle and where is it attained?

c) Determine the maximal curvature of the trajectory of the particle. Where is it attained? Draw the circle of curvature at that point.

d) Determine the asymptotic behavior of the curvature as $t \to \pm\infty$.

Remark: You may use the identity $\cosh^2 t - \sinh^2 t = 1$ without proof.

3. Evaluate:

a) $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$

b) $\displaystyle\int \frac{x+5}{x^3 - 2x^2 + x}\,\mathrm{d}x$

c) $\displaystyle\int x(x+1)^{40}\,\mathrm{d}x$

4. Determine the solution of the differential equation

$$\frac{1}{x} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{1+x^2} \cdot y(x) = 8$$

for $x > 0$ satisfying the initial condition $y(1) = 4$.

5. a) The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ converges because

- ◯ it is strictly increasing and bounded above by 3.
- ◯ it is strictly increasing and bounded below by 2.
- ◯ it is strictly decreasing and bounded above by 3.
- ◯ it is strictly decreasing and bounded below by 2.

b) Which one of the following statements about the behavior of a series is logically sound?

- ◯ The series has infinitely many terms greater than zero; therefore the series diverges.
- ◯ At each step we add less than in the previous one; therefore the series converges.
- ◯ The sequence of partial sums of the series is increasing; therefore the series converges.
- ◯ All terms of the series are positive and the series converges; therefore the series is absolutely convergent.

c) The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n 8^n}{n} x^{3n}$ is equal to:

- ◯ 0.
- ◯ $\frac{1}{2}$.
- ◯ $\frac{1}{8}$.
- ◯ 2.
- ◯ 8.
- ◯ ∞.

d) The domain of convergence of the power series $\sum_{n=0}^{\infty} \frac{(-1)^n 8^n}{n} x^{3n}$ is

- ○ $\mathbb{R}$
- ○ $(-\varrho, \varrho)$ for some $\varrho < \infty$.
- ○ $[-\varrho, \varrho)$ for some $\varrho < \infty$.
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- ○ $(-\varrho, \varrho]$ for some $\varrho < \infty$.

e) Which function is represented by the power series $\sum_{k=0}^{\infty} k(-2)^k x^k$ in its domain of convergence?

- ○ $(1 + 2x)^{-1}$
- ○ $(1 - 2x)^{-1}$
- ○ $-2 \cdot (1 + x)^{-2}$
- ○ $-2x \cdot (1 + 2x)^{-2}$
- ○ $-2x \cdot (1 - 2x)^{-2}$

6. Compute maximum and minimum of the function $f(x, y, z) = x + y - z$ on the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 3y^2 = 1,\ 4x = 3z\}.$$

7. Let $K(x, y, z) = (x, y, z)$ and F the surface

$$F = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$$

and n the unit normal vector of F pointing away from the origin.

Compute

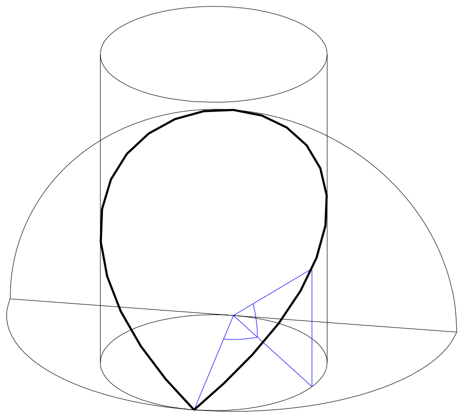
$$\int_F \operatorname{rot} K \cdot n \, \mathrm{d}o$$

a) directly.

b) using Stokes's theorem.

8. The Viviani window is the intersection of the upper hemisphere $x^2 + y^2 + z^2 = 4r^2$, $z \geq 0$ with the solid cylinder $(x - r)^2 + y^2 \leq r^2$, where $r > 0$.

Compute the surface area of the Viviani window.



9. Using the separation ansatz $u(x, y) = X(x)Y(y)$ determine the eigenvalues λ_{kl} and eigenfunctions u_{kl} of the eigenvalue problem

$$\Delta u + \lambda u = 0$$

on the rectangle $G = [0, 2\pi] \times [0, \pi]$ with Dirichlet boundary conditions $u = 0$ on ∂G.

10. Volume of the bicylinder.

The bicylinder D is the intersection of the two solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$, so

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq 1, y^2 + z^2 \leq 1\}.$$

a) The horizontal sections D_z, $-1 < z < 1$, all have the same shape. Which one?

- ○ A disk.
- ○ A square.
- ○ A cross.

b) For $-1 < z < 1$ the area $|D_z|$ of the horizontal section D_z is:

- ○ $|D_z| = \pi(1 - z^2)$
- ○ $|D_z| = \pi(1 - z)^2$
- ○ $|D_z| = 4(1 - z^2)$
- ○ $|D_z| = 2(1 - z)^2$

c) The volume of the bicylinder D is:

- ○ $|D| = \frac{4\pi}{3}$
- ○ $|D| = \frac{8\pi}{3}$
- ○ $|D| = \frac{16}{3}$
- ○ $|D| = 4$