Introduction to 3–manifolds

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Exercise 3.1. Let K be a knot in S^3. Compute all homology groups $H_i(X_K;\mathbb{Z})$ of the exterior X_K of K. Show that $H_1(X_K;\mathbb{Z})$ is generated by a meridian of the knot K.
Hint: Apply the Mayer-Vietoris sequence to the decomposition $S^3 = X_K \cup \big(S^1 \times D^2\big)$.

Exercise 3.2. Let a and b be integers.

1. Suppose that a and b are coprime. Show that there exists a diffeomorphism $f\colon T^2 \to T^2$, that sends the class $(a,b) \in \pi_1(T^2) = \mathbb{Z}^2$ to $(1,0)$.

   Hint: Construct a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ that descends to f.

2. Let $e\colon S^1 \to S^1 \times S^1$ be the map $z \mapsto \big(z^a, z^b\big)$. Prove that e is an embedding if and only if $\gcd(a,b) = 1$.

Exercise 3.3. For $p \geq 2$ and q coprime to p, consider the lens space $L(p,q)$, which is the quotient of $S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 2\}$ by the group action of $\mathbb{Z}_p$ given by $1 \cdot (z_1,z_2) = (e^{2\pi i/p} \cdot z_1, e^{2\pi i \cdot q/p} \cdot z_2)$. Denote the quotient map $S^3 \to L(p,q)$ by π, and define

$$T := \{(z_1,z_2) \in S^3 \colon |z_1|^2 = 1\} \cong S^1 \times S^1.$$

Show that $\pi(T) \subset L(p,q)$ is also a 2–torus, and determine $L(p,q)|\pi(T)$, the result of cutting $L(p,q)$ along $\pi(T)$.

Exercise 3.4. Two homeomorphisms $g_0, g_1\colon X \to X$ are *isotopic*, if there exists a continous map $F\colon X \times I \to X$ such that $f_t(x) := F(x,t)$ has the following properties: f_t is a homeomorphism for every $t \in I$, $f_0 = g_0$ and $f_1 = g_1$.

1. Prove that a homeomorphism $f\colon S^{n-1} \to S^{n-1}$ extends to a homeomorphism $F\colon D^n \to D^n$ for $n \geq 1$.

   Hint: The disk is homeomorphic to the cone $S^{n-1} \times [0,1]/\big(S^{n-1} \times \{1\}\big)$.

2. (Alexander trick) If a homemorphisms $f\colon D^n \to D^n$ restricts to the identity $\mathrm{id}_{S^{n-1}}$ on S^{n-1}, then f is isotopic to id_{D^n}. (Likely, you will also arrange that the isotopy f_t fulfills $f_t|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$ for every $t \in I$.)

   Hint: Let f happen in a smaller and smaller disk and extend by the identity.