**McMaster University** Fall 2017
**Faculty of Mathematics** Final Exam

# Algebraic Topology

## Math 731

Submission deadline: December 8, 2017 - 10:00 am

Examiner: Matthias Nagel

### Instructions

This exam comprises the cover page and two pages of 7 questions; make sure you have them all. Read the rest of this preamble carefully.

1. Each question is worth a score of 5 points. Full marks (25) are obtained by completely answering five questions. If you answer more than five, only the best five scores will be considered. Feel free to attempt all the questions.
2. Please answer the questions in the exam booklet provided. Show all work, and cite any results you use.
3. This a take-home exam. You are allowed to use your notes, and any textbooks or dictionaries.
4. Your are not allowed to seek or receive help from others. You may reach out to the instructor. You are not allowed to disclose the contents of the exam before the submission deadline.
5. Return the exam before Friday, December 8 - 10:00 am at my office HH M414.

## Problems

**Question 1** *Let $f\colon S^1 \to S^1$ be the map $z \mapsto z^3$, where $S^1 \subset \mathbb{C}$ denotes the unit circle. Let $X = S^1 \cup_f D^2$ be the result of attaching a 2-cell along $f$ to $S^1$. Compute the fundamental group $\pi_1(X)$.*
*Hint: Recall how attaching a 2–cell changes the fundamental group.*

**Question 2** *Compute the Euler characteristic of $S^1 \vee S^2$.*

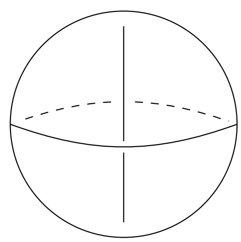
**Question 3** *Does $S^4 = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 : \|x\| = 1\}$ retract to its equator*

$$S^3 = \{(x_1, \ldots, x_5) \in S^4 : x_5 = 0\}?$$

**Question 4** *Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$, and consider the diameter*

$$d = \{(0, 0, t) \in \mathbb{R}^3\colon t \in [-1, 1]\}.$$

*Compute the homology groups of the subspace $X = S^2 \cup d$.*



**Question 5** *Consider the the chain complex $C$ of $\mathbb{Z}$–modules:*

$$C = \Big(0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix}} \mathbb{Z}/2 \xrightarrow{\phi} \mathbb{Z}/2 \to 0\Big),$$

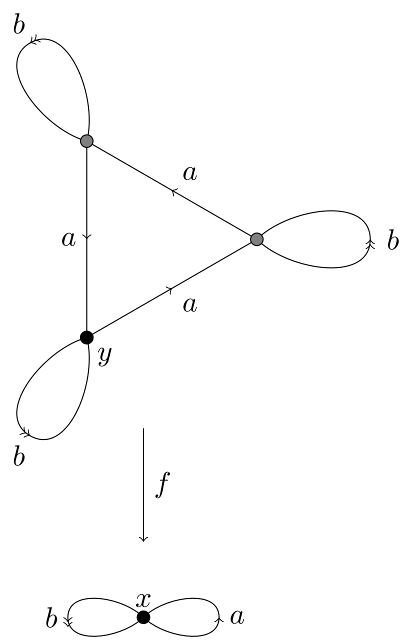
*where the rightmost $\mathbb{Z}/2$ has degree 0, and $\mathbb{Z}^m \xrightarrow{M} \mathbb{Z}^n$ denotes the map $v \mapsto Mv$ for a matrix $M \in \mathrm{Mat}(n \times m; \mathbb{Z})$.*
*Determine $\phi$, and verify that $C$ is a chain-complex. Compute the homology groups of the chain complex.*

**Question 6** *Let* $S^1$ *be the unit circle in* $\mathbb{R}^2$, *and*

$$W = \{(x, y) \in S^1 \colon x \leq 0\}$$

*its left half. Prove that the space* $X = S^1/W$ *resulting from squashing* $W$ *to a point is homeomorphic to* $S^1$.
*Hint: Write down a map* $S^1/W \to S^1$ *in polar coordinates, and recall under which circumstances a continous bijective map is automatically a homeomorphism.*

**Question 7** *Compute the image* $\pi_1(f)\big(\pi_1(Y, y)\big) \subset \pi_1(X, x) = F(\{a, b\})$ *of the the following cover* $f \colon Y \to X$. *The map* $f$ *sends edges to corresponding edges homeomorphically, while preserving the orientation.*




*Hint:* $Y$ *is a graph. Write down explicit generators for the fundamental group of* $Y$.