A 1.5-Approximation for Path TSP

Rico Zenklusen
ETH Zurich

Presentation: Martin Nägele, ETH Zurich
A brief intro to the Traveling Salesman Problem
The (Metric) Traveling Salesman Problem (TSP)

What’s the quickest way to visit $n$ sites?
Common variations of TSP

- **asymmetric**:
  - \( \begin{array}{c}
  7 \\
  2 
  \end{array} \)

- **distances between sites**

- **symmetric**:
  - \( \begin{array}{c}
  3 \\
  3 
  \end{array} \)

- Complete graph \( G = (V, E) \).
- Metric length \( \ell: E \rightarrow \mathbb{R}_{\geq 0} \).

- TSP:
  - Startpoint \( s \) and endpoint \( t \)

- **Path TSP**:
  - Startpoint \( s \) and endpoint \( t \)
- All variants are well-known to be APX-hard.

- Major open problem: what efficient computation can achieve.

<table>
<thead>
<tr>
<th>TSP</th>
<th>Path TSP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.667 [Hoogeveen, 1991]</td>
<td></td>
</tr>
<tr>
<td>1.618 [An, Kleinberg, Shmoys, 2012]</td>
<td></td>
</tr>
<tr>
<td>1.6 [Sebő, 2013]</td>
<td></td>
</tr>
<tr>
<td>1.5 [Christofides, 1978]</td>
<td>1.599 [Vygen, 2016]</td>
</tr>
<tr>
<td>1.5 + ( \varepsilon ) [Traub, Vygen, 2018a]</td>
<td></td>
</tr>
<tr>
<td>1.566 [Gottschalk, Vygen, 2016]</td>
<td></td>
</tr>
<tr>
<td>1.529 [Sebő, van Zuylen, 2016]</td>
<td></td>
</tr>
<tr>
<td>1.5 + ( \varepsilon ) [Traub, Vygen, 2018a]</td>
<td></td>
</tr>
</tbody>
</table>

Exciting progress for graph metrics:

[Oveis Gharan, Saberi, Singh, 2011]
[Mucha, 2014]
[Sebő, Vygen, 2014]
[Mömke, Svensson, 2016]
[Traub, Vygen, 2018b]
[...]
Our contribution

There is a 1.5-approximation for Path TSP.

▶ We move away from prior approaches, which focussed on so-called narrow cuts.

▶ Technical ingredients: Obtain a strong Held-Karp solution $z$ using
  ▶ Karger’s bound on the number of near-min cuts, and
  ▶ Dynamic programming “à la Traub & Vygen”.

Run a Christofides-type algorithm with a spanning tree obtained from $z$.

▶ Analysis follows Wolsey’s approach.

▶ Natural barrier 1.5: Any progress improves upon Christofides’ 1.5-approximation for TSP.
Following in Christofides’ footsteps

Why it works for TSP but fails for Path TSP…
(Spoiler: … and can be fixed.)
Find connected Eulerian graph with good total length, exploit metric lengths to shortcut.
Find connected Eulerian graph with good total length, exploit metric lengths to shortcut.

Start building a solution from a spanning tree
The general idea

- Find connected **Eulerian** graph with good total length, exploit metric lengths to shortcut.

  - Start building a solution from a **spanning tree**

  - Add **edges** to correct degree parities
The general idea

- Find connected Eulerian graph with good total length, exploit metric lengths to shortcut.

Start building a solution from a spanning tree

Add edges to correct degree parities

obtain Hamiltonian graph
The general idea

- Find connected Eulerian graph with good total length, exploit metric lengths to shortcut.

Start building a solution from a spanning tree

Add edges to correct degree parities

? obtain Hamiltonian graph
Christofides’ 1.5-approximation for TSP

1. Find a shortest spanning tree $T$.
   \[\Rightarrow \ell(T) \leq \ell(\text{OPT})\]

2. Find a shortest odd($T$)-join $J$.
   \[\Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT})\]

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   \[\Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT})\]
1. Find a shortest spanning tree $T$.
   $$\Rightarrow \ell(T) \leq \ell(\text{OPT}) .$$
2. Find a shortest odd-$T$-join $J$.
   $$\Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) .$$
3. Find Eulerian tour in multiunion of $T$ and $J$.
4. Return shortcutted Hamiltonian tour $H$.
   $$\Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) .$$
Christofides’ 1.5-approximation for TSP

1. Find a shortest spanning tree $T$.
   \[ \ell(T) \leq \ell(\text{OPT}) \]

2. Find a shortest odd($T$)-join $J$.
   \[ \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) \]

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   \[ \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) \]

\[ \text{odd}(T) := \{ v \in V \mid \deg(v) \text{ odd} \} \]
1. Find a shortest spanning tree $T$.\[ \implies \ell(T) \leq \ell(\text{OPT}). \]

2. Find a shortest $\text{odd}(T)$-join $J$.\[ \implies \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}). \]

3. Find Eulerian tour in multiunion of $T$ and $J$.\[ \implies \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}). \]

$\text{odd}(T) := \{ v \in V \mid \deg(v) \text{ odd} \}$
1. Find a shortest spanning tree $T$.
   $$\Rightarrow \ell(T) \leq \ell(\text{OPT}).$$

2. Find a shortest $\text{odd}(T)$-join $J$.
   $$\Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}).$$

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   $$\Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}).$$

$\text{odd}(T) := \{v \in V \mid \deg(v) \text{ odd}\}$
1. Find a shortest spanning tree $T$.
   \[ \ell(T) \leq \ell(\text{OPT}) \, . \]

2. Find a shortest $\text{odd}(T)$-join $J$.
   \[ \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) \, . \]

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   \[ \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) \, . \]

\[ \text{odd}(T) := \{ v \in V \mid \text{deg}(v) \text{ odd} \} \]
1. Find a shortest spanning tree $T$.
   \[ \Rightarrow \ell(T) \leq \ell(\text{OPT}) \]

2. Find a shortest odd($T$)-join $J$.
   \[ \Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) \]

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   \[ \Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) \]
Christofides’ 1.5-approximation for TSP

1. Find a shortest spanning tree \( T \).
   \[ \Rightarrow \ell(T) \leq \ell(\text{OPT}) \]

2. Find a shortest odd \( (T) \)-join \( J \).
   \[ \Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) \]

3. Find Eulerian tour in multiunion of \( T \) and \( J \).
4. Return shortcutted Hamiltonian tour \( H \).
   \[ \Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) \]
1. Find a shortest spanning tree $T$.
   \[ \Rightarrow \ell(T) \leq \ell(\text{OPT}) \]

2. Find a shortest odd($T$)-join $J$.
   \[ \Rightarrow \ell(J) \leq \frac{1}{2} \cdot \ell(\text{OPT}) \]

3. Find Eulerian tour in multiunion of $T$ and $J$.

4. Return shortcutted Hamiltonian tour $H$.
   \[ \Rightarrow \ell(H) \leq \ell(T) + \ell(J) \leq \frac{3}{2} \cdot \ell(\text{OPT}) \]
Held-Karp relaxation for TSP

- Held-Karp polytope

\[ P_{\text{HK}} := \left\{ x \in \mathbb{R}^E \geq 0 \ \left| \begin{array}{c}
  x(\delta(v)) = 2 \quad \forall v \in V \\
  x(\delta(C)) \geq 2 \quad \forall C \subsetneq V, \ C \neq \emptyset
\end{array} \right. \right\} . \]

- Held-Karp relaxation

\[ \min \{ \ell^T x \mid x \in P_{\text{HK}} \} . \]
Let $x^* \in \text{argmin}\{\ell^T x \mid x \in P_{HK}\}$.

$$P_{HK} = \left\{ x \in \mathbb{R}^E_\geq \left| \begin{array}{c} x(\delta(v)) = 2 \quad \forall v \in V \\ x(\delta(C)) \geq 2 \quad \forall C \subseteq V, C \neq \emptyset \end{array} \right. \right\}$$

**Claim**

If $T$ is a shortest spanning tree, and $J$ is a shortest $\text{odd}(T)$-join, then

1. $\ell(T) \leq \ell^T x^*$, and
2. $\ell(J) \leq \frac{1}{2} \cdot \ell^T x^*$.
Let $x^* \in \text{argmin}\{\ell^T x \mid x \in P_{\text{HK}}\}$.

$$
P_{\text{HK}} = \left\{ x \in \mathbb{R}_{\geq 0}^E \left| \begin{array}{l} x(\delta(v)) = 2 \quad \forall v \in V \\ x(\delta(C)) \geq 2 \quad \forall C \subseteq V, C \neq \emptyset \end{array} \right. \right\}
$$

### Claim

If $T$ is a shortest spanning tree, and $J$ is a shortest odd($T$)-join, then

1. (a) $\ell(T) \leq \ell^T x^*$, and

2. (b) $\ell(J) \leq \frac{1}{2} \cdot \ell^T x^*$.

#### (a) $\frac{n-1}{n} \cdot x^* \in P_{\text{ST}}$.

$$
P_{\text{ST}} = \left\{ x \in \mathbb{R}_{\geq 0}^E \left| \begin{array}{l} x(E) = |V| - 1 \\ x(E[S]) \leq |S| - 1 \quad \forall S \subseteq V, S \neq \emptyset \end{array} \right. \right\}
$$
Let \( x^* \in \text{argmin}\{\ell^T x \mid x \in P_{HK}\} \).

\[
P_{HK} = \left\{ x \in \mathbb{R}_{\geq 0}^E \left| \begin{array}{l}
x(\delta(v)) = 2 \quad \forall v \in V \\
x(\delta(C)) \geq 2 \quad \forall C \subset V, C \neq \emptyset
\end{array} \right. \right\}
\]

**Claim**

If \( T \) is a shortest spanning tree, and \( J \) is a shortest odd(\( T \))-join, then

\[
(a) \quad \ell(T) \leq \ell^T x^*, \quad \text{and} \quad (b) \quad \ell(J) \leq \frac{1}{2} \cdot \ell^T x^*.
\]

\( (a) \quad \frac{n-1}{n} \cdot x^* \in P_{ST}. \)

\[
P_{ST} = \left\{ x \in \mathbb{R}_{\geq 0}^E \left| \begin{array}{l}
x(E) = |V| - 1 \\
x(E[S]) \leq |S| - 1 \quad \forall S \subset V, S \neq \emptyset
\end{array} \right. \right\}
\]

\( (b) \quad \frac{1}{2} \cdot x^* \in P_{Q\text{-join}}^\uparrow \)

for any \( Q \subset V, |Q| \) even.
Let \( x^* \in \arg \min \{ \ell^T x \mid x \in P_{HK} \} \).

\[
P_{HK} = \left\{ x \in \mathbb{R}^E_{\geq 0} \left| \begin{array}{c}
x(\delta(v)) = 2 \quad \forall v \in V \\
x(\delta(C)) \geq 2 \quad \forall C \subseteq V, C \neq \emptyset
\end{array} \right. \right\}
\]

**Claim**

If \( T \) is a shortest spanning tree, and \( J \) is a shortest odd(\( T \))-join, then

\[(a) \quad \ell(T) \leq \ell^T x^*, \quad \text{and} \quad (b) \quad \ell(J) \leq \frac{1}{2} \cdot \ell^T x^*.
\]

\[(a) \quad \frac{n-1}{n} \cdot x^* \in P_{ST}.
\]

\[
P_{ST} = \left\{ x \in \mathbb{R}^E_{\geq 0} \left| \begin{array}{c}
x(E) = |V| - 1 \\
x(E[S]) \leq |S| - 1 \quad \forall S \subseteq V, S \neq \emptyset
\end{array} \right. \right\}
\]

\[(b) \quad \frac{1}{2} \cdot x^* \in P_{Q-join}^\uparrow
\]

for any \( Q \subseteq V, |Q| \) even.

\[
P_{Q-join}^\uparrow = \left\{ x \in \mathbb{R}^E_{\geq 0} \left| x(\delta(C)) \geq 1 \quad \forall C \subseteq V, |C \cap Q| \text{ odd} \right. \right\}
\]

Shows 1.5-approximation and upper bound on integrality gap.
Christofides’ approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(\text{OPT}).$

- But: OPT does not contain two disjoint $Q_T$-joins.

- Still, shortest $Q_T$-join $J$ satisfies
  \[ \ell(J) \leq \frac{2}{3} \cdot \ell(\text{OPT}). \]  
  [Hoogeveen, 1991]

  \textbf{Proof:} Together, OPT and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.
Christofides’ approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(\text{OPT})$.
- But: OPT does not contain two disjoint $Q_T$-joins.
- Still, shortest $Q_T$-join $J$ satisfies
  \[ \ell(J) \leq \frac{2}{3} \cdot \ell(\text{OPT}). \]  
  [Hoogeveen, 1991]

**Proof:** Together, OPT and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.
Christofides’ approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(\text{OPT})$.
- But: OPT does not contain two disjoint $Q_T$-joins.
- Still, shortest $Q_T$-join $J$ satisfies
  \[ \ell(J) \leq \frac{2}{3} \cdot \ell(\text{OPT}). \]  
  \[\text{[Hoogeveen, 1991]}\]

**Proof:** Together, OPT and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.

$Q_T := \text{odd}(T) \triangle \{s, t\}$
Christofides’ approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(\text{OPT})$.

- But: OPT does not contain two disjoint $Q_T$-joins.

- Still, shortest $Q_T$-join $J$ satisfies
  \[ \ell(J) \leq \frac{2}{3} \cdot \ell(\text{OPT}). \]  
  [Hoogeveen, 1991]

**Proof:** Together, OPT and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.

$Q_T := \text{odd}(T) \triangle \{s, t\}$
Christofides’ approach for Path TSP

[Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(\text{OPT}).$

- But: OPT does not contain two disjoint $Q_T$-joins.

- Still, shortest $Q_T$-join $J$ satisfies
  $$\ell(J) \leq \frac{2}{3} \cdot \ell(\text{OPT}).$$

  [Hoogeveen, 1991]

  \textbf{Proof:} Together, OPT and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.

\[ Q_T := \text{odd}(T) \triangle \{s, t\} \]
Christofides’ approach for Path TSP [Hoogeveen, 1991]

- Shortest spanning tree $T$: $\ell(T) \leq \ell(OPT)$.

- But: $OPT$ does not contain two disjoint $Q_T$-joins.

- Still, shortest $Q_T$-join $J$ satisfies
  \[ \ell(J) \leq \frac{2}{3} \cdot \ell(OPT). \]

  [Hoogeveen, 1991]

**Proof:** Together, $OPT$ and $T$ contain three $Q_T$-joins.

- This algorithm is only $\frac{5}{3}$-approximate on some instances.

Goal: Find tree $T$ with $\ell(T) \leq \ell(OPT)$ and s.t. shortest $Q_T$-join $J$ satisfies $\ell(J) \leq \frac{1}{2} \cdot \ell(OPT)$. 
Held-Karp polytope for Path TSP:

\[ P_{HK} := \left\{ x \in \mathbb{R}^E_{\geq 0} \mid \begin{array}{l}
    x(\delta(v)) = 1 \quad v \in \{s, t\} \\
    x(\delta(v)) = 2 \quad v \in V \setminus \{s, t\} \\
    x(\delta(C)) \geq 1 \quad \forall C \subseteq V, \, |C \cap \{s, t\}| = 1 \\
    x(\delta(C)) \geq 2 \quad \forall C \subseteq V, \, C \neq \emptyset, \, |C \cap \{s, t\}| = 0
\end{array} \right\} \]
Held-Karp polytope for Path TSP:

\[ P_{HK} := \left\{ x \in \mathbb{R}^E_{\geq 0} \right. \right| \begin{align*}
  \left| x(\delta(v)) \right| &= 1 \quad v \in \{s, t\} \\
  \left| x(\delta(v)) \right| &= 2 \quad v \in V \setminus \{s, t\} \\
  x(\delta(C)) &\geq 1 \quad \forall C \subseteq V, \ |C \cap \{s, t\}| = 1 \\
  x(\delta(C)) &> 1 \quad \forall C \subseteq V, \ C \neq \emptyset, \ |C \cap \{s, t\}| = 0
\end{align*} \]

- \( x^*(e) = \frac{1}{3} \) for \( e \) that forms a chain.
- \( x^*(e) = \frac{2}{3} \) for edges with odd cuts that do not form a chain.

\[ P_{HK}^{\uparrow Q_T} = \{ x \in \mathbb{R}^E_{\geq 0} \right| \begin{align*}
  \left| x(\delta(C)) \right| \geq 1 \quad \forall C \subseteq V, \ |C \cap Q_T| \text{ odd}
\end{align*} \]
Where Wolsey’s analysis fails

Held-Karp polytope for Path TSP:

\[
P_{HK} := \left\{ x \in \mathbb{R}^E_{\geq 0} \left| \begin{array}{l}
x(\delta(v)) = 1 \quad v \in \{s, t\} \\
x(\delta(v)) = 2 \quad v \in V \setminus \{s, t\} \\
x(\delta(C)) \geq 1 \quad \forall C \subseteq V, |C \cap \{s, t\}| = 1 \\
x(\delta(C)) \geq 2 \quad \forall C \subsetneq V, C \neq \emptyset, |C \cap \{s, t\}| = 0
\end{array} \right. \right\}
\]

Problem: \( \frac{x^*}{2} \) for \( x^* \in \arg\min \{ \ell^T x \mid x \in P_{HK} \} \) infeasible for \( P_{QT}-join \).

\[
P_{QT}-join = \left\{ x \in \mathbb{R}^E_{\geq 0} \left| x(\delta(C)) \geq 1 \quad \forall C \subseteq V, \left| C \cap Q_T \right| \text{odd} \right. \right\}
\]
Where Wolsey’s analysis fails

Held-Karp polytope for Path TSP:

\[
P_{HK} := \left\{ x \in \mathbb{R}^E_{\geq 0} \left| \begin{array}{ll}
x(\delta(v)) = 1 & v \in \{s, t\} \\
x(\delta(v)) = 2 & v \in V \setminus \{s, t\} \\
x(\delta(C)) \geq 1 & \forall C \subseteq V, |C \cap \{s, t\}| = 1 \\
x(\delta(C)) \geq 2 & \forall C \supseteq V, C \neq \emptyset, |C \cap \{s, t\}| = 0 \end{array} \right. \right\}
\]

Problem: \( \frac{x^*}{2} \) for \( x^* \in \text{argmin} \{ \ell^T x \mid x \in P_{HK} \} \) infeasible for \( P_{Q_T-join}^\uparrow \).

\[
P_{Q_T-join}^\uparrow = \left\{ x \in \mathbb{R}^E_{\geq 0} \left| x(\delta(C)) \geq 1 \forall C \subseteq V, \left| C \cap Q_T \right| \text{ odd} \right. \right\}
\]

Infeasibility caused by narrow cuts:

→ cuts \( C \) with \( x^*(\delta(C)) < 2 \).
→ s-t-cuts, form a chain.
→ appear in \( P_{Q_T-join}^\uparrow \) only if \( |T \cap \delta(C)| \) even.
Where Wolsey’s analysis fails

**Held-Karp polytope for Path TSP:**

\[
P_{HK} := \left\{ x \in \mathbb{R}^E_{\geq 0} \middle| \begin{array}{l}
    x(\delta(v)) = 1 \quad \forall v \in \{s, t\} \\
    x(\delta(v)) = 2 \quad \forall v \in V \setminus \{s, t\} \\
    x(\delta(C)) \geq 1 \quad \forall C \subseteq V, |C \cap \{s, t\}| = 1 \\
    x(\delta(C)) \geq 2 \quad \forall C \subsetneq V, C \neq \emptyset, |C \cap \{s, t\}| = 0
\end{array} \right\}
\]

**Problem:** \(\frac{x^*}{2}\) for \(x^* \in \arg\min \{\ell^T x \mid x \in P_{HK}\}\) infeasible for \(P_{QT-join}^\uparrow\).

\[
P_{QT-join}^\uparrow = \left\{ x \in \mathbb{R}^E_{\geq 0} \middle| x(\delta(C)) \geq 1 \quad \forall C \subseteq V, |C \cap QT| \text{ odd} \right\}
\]

**Infeasibility caused by narrow cuts:**

→ cuts \(C\) with \(x^*(\delta(C)) < 2\).
→ \(s\)-\(t\)-cuts, form a chain.
→ appear in \(P_{QT-join}^\uparrow\) only if \(|T \cap \delta(C)|\) even.
1.5-approximation: The high-level plan
Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{\text{HK}}\}$.

Let

$$B(x^*) := \{ C \subseteq V \mid s \in C, \ t \notin C, \ x^*(\delta(C)) < 3 \}.$$  

By Karger's result, $|B(x^*)|$ is polynomially bounded. [Karger 1993]

We will find a shortest point $y \in P_{\text{HK}}$ that is $B(x^*)$-good:

For each $B \in B(x^*)$, either

- $y(\delta(B)) \geq 3$, or
- $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, $\uparrow y(\delta(B)) \geq 3$, or $\uparrow y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

$x^* \in P_{HK}$

$\mathcal{B}(x^*)$-good point $y$

$x^*(e) = \frac{1}{3}$

$x^*(e) = \frac{2}{3}$

$x^*(e) = 1$
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

$x^* \in P_{HK}$

$\mathcal{B}(x^*)$-good point $y$
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, $y(\delta(B)) \geq 3$, or $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

$x^*(e) = \frac{1}{3}$

$x^*(e) = \frac{2}{3}$

$x^*(e) = 1$

$x^* \in P_{HK}$

$\mathcal{B}(x^*)$-good point $y$
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

\begin{itemize}
  \item $x^*(e) = \frac{1}{3}$
  \item $x^*(e) = \frac{2}{3}$
  \item $x^*(e) = 1$
\end{itemize}
A new ingredient: Finding $\mathcal{B}(x^\ast)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^\ast)$-good: For all $B \in \mathcal{B}(x^\ast)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

$x^\ast \in P_{HK}$

$\mathcal{B}(x^\ast)$-good point $y$

---

- $x^\ast(e) = \frac{1}{3}$
- $x^\ast(e) = \frac{2}{3}$
- $x^\ast(e) = 1$
A new ingredient: Finding \( \mathcal{B}(x^*) \)-good points

\( y \in P_{HK} \) is \( \mathcal{B}(x^*) \)-good: For all \( B \in \mathcal{B}(x^*) \), \( y(\delta(B)) \geq 3 \), or \( y(\delta(B)) = 1 \) and \( y \) is 0/1 on \( \delta(B) \).

\[ x^*(e) = \frac{1}{3} \]
\[ x^*(e) = \frac{2}{3} \]
\[ x^*(e) = 1 \]
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$. 

$B(x^*)$-good

Theorem

Let $\mathcal{B} \subseteq 2^\mathcal{V}$ be family of $s$-$t$ cuts. A shortest $\mathcal{B}(x^*)$-good point $y \in P_{HK}$ can be found in time $O(\text{poly}(|\mathcal{V}|,|\mathcal{B}|))$. 

\begin{itemize}
  \item $x^*(e) = 1/3$
  \item $x^*(e) = 2/3$
  \item $x^*(e) = 1$
\end{itemize}

$B(x^*)$-good point $y$
A new ingredient: Finding $\mathcal{B}(x^*)$-good points

$\mathcal{B}(x^*)$-good

$y \in P_{HK}$ is $\mathcal{B}(x^*)$-good: For all $B \in \mathcal{B}(x^*)$, ▶ $y(\delta(B)) \geq 3$, or ▶ $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

Theorem

Let $\mathcal{B} \subseteq 2^V$ be family of s-t cuts. A shortest $\mathcal{B}$-good point $y \in P_{HK}$ can be found in time $O(\text{poly}(|V|, |\mathcal{B}|))$. 

\[ x^*(e) = \frac{1}{3} \]
\[ x^*(e) = \frac{2}{3} \]
\[ x^*(e) = 1 \]
From short $B(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \arg\min\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.
From short $B(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \arg\min\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 

\[ x^*(e) = \begin{cases} 1/3 \\ 2/3 \\ 1 \end{cases} \]
From short $B(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \text{argmin} \{ \ell(x) \mid x \in P_{HK} \}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

\[
\begin{align*}
\dash \cdots \quad x^*(e) &= \frac{1}{3} \\
\cdash \cdash \quad x^*(e) &= \frac{2}{3} \\
\quad \quad \quad x^*(e) &= 1
\end{align*}
\]
From short $B(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 

\[ Q_T := \text{odd}(T) \triangle \{s, t\} \]
From short $\mathcal{B}(x^*)$-good points to $1.5$-approx.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{\text{HK}}\}$.
2. Let $y$ be a shortest $\mathcal{B}(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 
From short $\mathcal{B}(x^*)$-good points to 1.5-approx.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $\mathcal{B}(x^*)$-good point.
3. Let $T$ be a shortest spanning tree in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 

Diagram: [Diagram of a graph with nodes and edges, indicating a path from $s$ to $t$]
The spanning tree $T$ is cheap: $\ell(T) \leq \ell(OPT)$

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 

We have $y \in P_{HK} \subseteq P_{ST}$. 

$\Rightarrow \ell(T) \leq \ell^{\top}y$. 

$OPT$ is $B$-good for any family $B$ of $s$-$t$ cuts. 

$\Rightarrow \ell^{\top}y \leq \ell(OPT)$. 

Together, we conclude $\ell(T) \leq \ell^{\top}y \leq \ell(OPT)$.
The spanning tree $T$ is cheap: $\ell(T) \leq \ell(\text{OPT})$

We have $y \in P_{\text{HK}} \subseteq P_{\text{ST}}$.

$\implies \ell(T) \leq \ell^\top y$.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{\text{HK}}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 
The spanning tree $T$ is cheap: $\ell(T) \leq \ell(\text{OPT})$

1. Let $x^* \in \arg\min \{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

We have $y \in P_{HK} \subseteq P_{ST}$.

$\implies \ell(T) \leq \ell^T y$.

OPT is $B$-good for any family $B$ of $s$-$t$ cuts.

$\implies \ell^T y \leq \ell(\text{OPT})$. 

Together, we conclude \(\ell(T) \leq \ell^T y \leq \ell(\text{OPT})\).
The spanning tree $T$ is cheap: $\ell(T) \leq \ell(\text{OPT})$.

1. Let $x^* \in \arg\min \{ \ell(x) \mid x \in P_{HK} \}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

- We have $y \in P_{HK} \subseteq P_{ST}$.
  $\implies \ell(T) \leq \ell^T y$.

- OPT is $B$-good for any family $B$ of $s$-$t$ cuts.
  $\implies \ell^T y \leq \ell(\text{OPT})$.

- Together, we conclude
  $\ell(T) \leq \ell^T y \leq \ell(\text{OPT})$.
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$.

- We show $\frac{1}{4} x^* + \frac{1}{4} y \in P_{Q_T}^\uparrow$.

$$\implies \ell(J) \leq \frac{1}{4} (\ell^\top x^* + \ell^\top y) \leq \frac{1}{2} \ell(\text{OPT}).$$

- Distinguish cases:

1. Let $x^* \in \arg\min \{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

\[ P_{Q_T}^\uparrow = \left\{ x \in \mathbb{R}_E^+ \mid x(\delta(C)) \geq 1 \quad \forall C \subseteq V, \ |C \cap Q_T| \text{ odd} \right\} \]
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$

1. Let $x^* \in \arg\min\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

$P_{Q_T}^\uparrow = \{ x \in \mathbb{R}^E_{\geq 0} \mid x(\delta(C)) \geq 1 \ \forall C \subseteq V, |C \cap Q_T| \text{ odd} \}$

- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P_{Q_T}^\uparrow$.
  $$\implies \ell(J) \leq \frac{1}{4} (\ell^\top x^* + \ell^\top y) \leq \frac{1}{2} \ell(\text{OPT}) \ .$$

Distinguish cases:

1. Non $s$-$t$ cuts.
2. $x^* \in P_{HK}$
3. $x^*(\delta(B)) \geq 2$
4. $y(\delta(B)) \geq 2$
5. $B(x^*)$-good point $y$
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in \mathcal{P}_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$.

We show $\frac{1}{4} x^* + \frac{1}{4} y \in P_{Q_T}$-join.

$$\implies \ell(J) \leq \frac{1}{4} (\ell^\top x^* + \ell^\top y) \leq \frac{1}{2} \ell(\text{OPT}).$$

Distinguish cases:

1. $s$-$t$ cuts not in $B(x^*)$.
2. $y$ is a shortest $B(x^*)$-good point.
3. $y((\delta(B))) \geq 1$
4. $|B \cap Q_T| \text{ even}$

$P_{Q_T}$-join = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid x((\delta(C)) \geq 1 \forall C \subseteq V, |C \cap Q_T| \text{ odd} \right\}$
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$

- We show $\frac{1}{4} x^* + \frac{1}{4} y \in P_{Q_T}^\uparrow$.

  $$\implies \ell(J) \leq \frac{1}{4} (\ell^\top x^* + \ell^\top y) \leq \frac{1}{2} \ell(\text{OPT}).$$

- Distinguish cases:
  1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{HK}\}$.
  2. Let $y$ be a shortest $B(x^*)$-good point.
  3. Let $T$ be an MST in $(V, \text{supp}(y))$.
  4. Let $J$ be a shortest $Q_T$-join.
  5. Return shortcutted tour in multiunion of $T$ and $J$.

\[ P_{Q_T}^\uparrow = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid x(\delta(C)) \geq 1 \quad \forall C \subseteq V, \ |C \cap Q_T| \text{ odd} \right\} \]
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$

- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P_{Q_T}^{\uparrow}$. 

  $\implies \ell(J) \leq \frac{1}{4}(\ell^T x^* + \ell^T y) \leq \frac{1}{2} \ell(\text{OPT})$.

- Distinguish cases:
  1. $1$-s-t cuts $B \in B(x^*)$ with $y(\delta(B)) = 1$.

We show $\frac{1}{4}x^* + \frac{1}{4}y \in P_{Q_T}^{\uparrow}$.

1. Let $x^* \in \text{argmin}\{\ell(x) \mid x \in P_{HK}\}$.
2. Let $y$ be a shortest $B(x^*)$-good point.
3. Let $T$ be an MST in $(V, \text{supp}(y))$.
4. Let $J$ be a shortest $Q_T$-join.
5. Return shortcutted tour in multiunion of $T$ and $J$. 

$P_{Q_T}^{\uparrow} = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid x(\delta(C)) \geq 1 \quad \forall C \subseteq V, \ |C \cap Q_T| \text{ odd} \right\}$

$s$-t cuts $B \in B(x^*)$ with $y(\delta(B)) = 1$.

$x^* \in P_{HK}$

$x^*(\delta(B)) \geq 1 \quad y(\delta(B)) = 1$
The $Q_T$-join $J$ is cheap: $\ell(J) \leq \frac{1}{2} \ell(\text{OPT})$

- We show $\frac{1}{4}x^* + \frac{1}{4}y \in P_{Q_T}$-join.
  \[ \implies \ell(J) \leq \frac{1}{4} (\ell^T x^* + \ell^T y) \leq \frac{1}{2} \ell(\text{OPT}). \]

- Distinguish cases:
  1. Let $x^* \in \arg\min\{\ell(x) \mid x \in P_{\text{HK}}\}$.
  2. Let $y$ be a shortest $B(x^*)$-good point.
  3. Let $T$ be an MST in $(V, \text{supp}(y))$.
  4. Let $J$ be a shortest $Q_T$-join.
  5. Return shortcutted tour in multiunion of $T$ and $J$.

$P_{Q_T} = \left\{ x \in \mathbb{R}^E_{\geq 0} \mid x(\delta(C)) \geq 1 \quad \forall C \subseteq V, |C \cap Q_T| \text{ odd} \right\}$
The dynamic program
## Theorem

Let $B \subseteq 2^V$ a family of $s$-$t$ cuts. A shortest $B$-good point $y \in P_{HK}$ can be found in time $O(\text{poly}(|V|, |B|))$.

For all $B \in B$, either
- $y(\delta(B)) \geq 3$, or
- $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.
Theorem

Let $\mathcal{B} \subseteq 2^V$ a family of $s$-$t$ cuts. A shortest $\mathcal{B}$-good point $y \in P_{HK}$ can be found in time $O(\text{poly}(|V|, |\mathcal{B}|))$.

For all $B \in \mathcal{B}$, either
- $y(\delta(B)) \geq 3$, or
- $y(\delta(B)) = 1$ and $y$ is 0/1 on $\delta(B)$.

- DP can be interpreted as a simplified version of the one used by Traub & Vygen [SODA 2018].
- Key plan:
  - “Guess” cuts $B_1, \ldots, B_k \in \mathcal{B}$ with $y(\delta(B_i)) = 1$, and the single edge in these cuts.
  - Observation: $B_1, \ldots, B_k$ must form a chain $\rightarrow$ can split into subproblems on $B_{i+1} \setminus B_i$. 
The DP: Finding shortest \( B \)-good points

**Theorem**

Let \( B \subseteq 2^V \) a family of \( s-t \) cuts. A shortest \( B \)-good point \( y \in P_{HK} \) can be found in time \( O(\text{poly}(|V|, |B|)) \).

- DP can be interpreted as a simplified version of the one used by Traub & Vygen [SODA 2018].
- Key plan:
  - “Guess” cuts \( B_1, \ldots, B_k \in B \) with \( y(\delta(B_i)) = 1 \), and the single edge in these cuts.
  - Observation: \( B_1, \ldots, B_k \) must form a chain \( \rightarrow \) can split into subproblems on \( B_{i+1} \setminus B_i \).

For all \( B \in B \), either
- \( y(\delta(B)) \geq 3 \), or
- \( y(\delta(B)) = 1 \) and \( y \) is 0/1 on \( \delta(B) \).
Solving a single subproblem

- Restriction to $B_{i+1} \setminus B_i$, start at $u_i$, end at $v_{i+1}$.

- Enforce $y(\delta(B)) \geq 3$ for $B \in \mathcal{B}$ with $B_i \subsetneq B \subsetneq B_{i+1}$.

- Corresponding LP formulation:

$$\lambda(B_{i+1} \setminus B_i, u_i, v_{i+1}) = \min \ell^\top y$$
$$y \in P_{\text{HK}}(B_{i+1} \setminus B_i, u_i, v_{i+1})$$
$$y(\delta(B)) \geq 3 \quad \forall B \in \mathcal{B}: B_i \subsetneq B \subsetneq B_{i+1}.$$
Solving a single subproblem

- Restriction to $B_{i+1} \setminus B_i$, start at $u_i$, end at $v_{i+1}$.
- Enforce $y(\delta(B)) \geq 3$ for $B \in \mathcal{B}$ with $B_i \subset B \subset B_{i+1}$.
- Corresponding LP formulation:

\[
\lambda(B_{i+1} \setminus B_i, u_i, v_{i+1}) = \min \ell^\top y
\]

\[
y \in P_{HK}(B_{i+1} \setminus B_i, u_i, v_{i+1})
\]

\[
y(\delta(B)) \geq 3 \quad \forall B \in \mathcal{B}: B_i \subset B \subset B_{i+1}
\]
Restricion to $B_{i+1} \setminus B_i$, start at $u_i$, end at $v_{i+1}$.

Enforce $y(\delta(B)) \geq 3$ for $B \in \mathcal{B}$ with $B_i \subsetneq B \subsetneq B_{i+1}$.

Corresponding LP formulation:

$$\lambda(B_{i+1} \setminus B_i, u_i, v_{i+1}) = \min \ell^\top y$$

$y \in P_{HK}(B_{i+1} \setminus B_i, u_i, v_{i+1})$

$y(\delta(B)) \geq 3 \quad \forall B \in \mathcal{B}: B_i \subsetneq B \subsetneq B_{i+1}$.
Idea: Advance from one cut $B$ with $y(\delta(B)) = 1$ to another.
Setting up the DP

- Idea: Advance from one cut $B$ with $y(\delta(B)) = 1$ to another.
- Formulation as a shortest path problem on auxiliary digraph:

  **Nodes:** Pairs $(B, v)$ for $B \in \mathcal{B}$ and $v \in V$.

  **Edges:** Two types of steps corresponding to extension of a solution.

Optimal solution: Shortest $(\{s\}, s) - (V\{t\}, t)$ path in auxiliary digraph.
Setting up the DP

- Idea: Advance from one cut $B$ with $y(\delta(B)) = 1$ to another.

- Formulation as a shortest path problem on auxiliary digraph:
  **Nodes:** Pairs $(B, v)$ for $B \in \mathcal{B}$ and $v \in V$.
  **Edges:** Two types of steps corresponding to extension of a solution.

- Optimal solution: Shortest $(\{s\}, s) - (V \setminus \{t\}, t)$ path in auxiliary digraph.
DP auxiliary graph: An example

\[ B_1 = \{s\} \]

\[ B' \]

\[ B_2 \]

\[ B_2 \]

\[ B_3 \]

\[ B_3 \]

\[ B_4 = V \setminus \{t\} \]
DP auxiliary graph: An example

$B_1 = \{s\}$

$B_2 = B'$

$B_3 = B_2$,

$B_4 = V \setminus \{t\}$
DP auxiliary graph: An example

\[ B_1 = \{ s \} \]

\[ B' = \{ \{ s \}, u' \} \]

\[ B_2 = \{ s \}, v_2 \}

\[ B_3 = \{ v_3 \}, u_3 \}

\[ B_4 = V \setminus \{ t \} \]

\[ \{ s \}, u_1 \]

\[ \{ s \}, s \]

\[ \{ s \}, u' \]
DP auxiliary graph: An example

\[ B_1 = \{s\} \]

\[ B' \]

\[ B_2 \]

\[ B_3 \]

\[ B_4 = V \setminus \{t\} \]
DP auxiliary graph: An example

\( B_1 = \{s\} \)
\( B' \)
\( B_2 \)
\( B_3 \)
\( B_4 = V \setminus \{t\} \)
DP auxiliary graph: An example

\[ B_1 = \{s\} \]

\[ B' = v_2 \]

\[ B_2 = v_2 \]

\[ B_3 = v_3 \]

\[ B_4 = V \setminus \{t\} \]
DP auxiliary graph: An example
DP auxiliary graph: An example

- $B_1 = \{s\}$
- $B' = \{s\}$
- $B_2 = \{t\}$
- $B_3 = \{t\}$
- $B_4 = V \setminus \{t\}$
DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.
DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

**Theorem (basic properties of DP solutions)**

- Any DP solution is in $P_{HK}$.
- Any DP solution is $B$-good.
- Shortest DP solution has length at most $\ell(OPT)$. 
DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

**Theorem (basic properties of DP solutions)**

- Any DP solution is in $P_{HK}$.
- Any DP solution is $B$-good.
- Shortest DP solution has length at most $\ell(\text{OPT})$.

**Proof.** Need to show that DP solution $y$ satisfies $y(\delta(B)) \geq 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$.
Getting solutions through DP

- DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

### Theorem (basic properties of DP solutions)

- Any DP solution is in $P_{HK}$.
- Any DP solution is $B$-good.
- Shortest DP solution has length at most $\ell(OPT)$.

**Proof.** Need to show that DP solution $y$ satisfies $y(\delta(B)) \geq 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$. 

![Diagram showing cuts $B_i$ and $B$]
DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

**Theorem (basic properties of DP solutions)**

- Any DP solution is in $P_{HK}$.
- Any DP solution is $B$-good.
- Shortest DP solution has length at most $\ell(OPT)$.

**Proof.** Need to show that DP solution $y$ satisfies $y(\delta(B)) \geq 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$.

$$y(\delta(B)) + y(\delta(B_i))$$
Getting solutions through DP

- DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

<table>
<thead>
<tr>
<th>Theorem (basic properties of DP solutions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Any DP solution is in $P_{HK}$.</td>
</tr>
<tr>
<td>- Any DP solution is $B$-good.</td>
</tr>
<tr>
<td>- Shortest DP solution has length at most $\ell(\text{OPT})$.</td>
</tr>
</tbody>
</table>

Proof. Need to show that DP solution $y$ satisfies $y(\delta(B)) \geq 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$.

\[ y(\delta(B)) + y(\delta(B_i)) \geq y(\delta(B_i \setminus B)) + y(\delta(B \setminus B_i)) \]
Getting solutions through DP

- DP solution: Combination of edges in cuts $B_i$ and partial Held-Karp solutions.

**Theorem (basic properties of DP solutions)**

- Any DP solution is in $P_{HK}$.  
- Any DP solution is $B$-good.  
- Shortest DP solution has length at most $\ell(OPT)$.

**Proof.** Need to show that DP solution $y$ satisfies $y(\delta(B)) \geq 3$ for all $B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\}$.

$$y(\delta(B)) + y(\delta(B_i)) \geq y(\delta(B_i \setminus B)) + y(\delta(B \setminus B_i))$$
Getting solutions through DP

- DP solution: Combination of edges in cuts \( B_i \) and partial Held-Karp solutions.

**Theorem (basic properties of DP solutions)**

- Any DP solution is in \( P_{HK} \).
- Any DP solution is \( \mathcal{B} \)-good.
- Shortest DP solution has length at most \( \ell(OPT) \).

**Proof.** Need to show that DP solution \( y \) satisfies \( y(\delta(B)) \geq 3 \) for all \( B \in \mathcal{B} \setminus \{B_1, \ldots, B_k\} \).

\[
y(\delta(B)) + y(\delta(B_i)) \geq y(\delta(B_i \setminus B)) + y(\delta(B \setminus B_i))
\]

\[
= 1 \quad \geq 2 \quad \geq 2
\]

\[
\implies y(\delta(B)) \geq 3.
\]
Conclusions
Theorem [Zenklusen, 2018]

There is a 1.5-approximation for Path TSP.

- Approximation factors below 1.5 for TSP (or even Path TSP)?
- Show that the integrality gap of Held-Karp relaxation for Path TSP is 1.5.
  
  \textit{Current best: 1.5284.} [Traub, Vygen, 2018c]

- 1.5-approximation for $T$-tours?
  
  \textit{True if $|T| = O(1)$.} [N., Zenklusen, 2019]