Bipartite Ramsey number of paths and cycles

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- Rosta (1973) and Faudree, Schelp (1974):

$$R(C_n) = \begin{cases} \frac{3}{2}n - 1, \text{for } n \text{ even} \\ 2n - 1, \text{for } n \text{ odd} \end{cases}$$

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- For $k \geq 3$ what is: $BR_k(P_n) = ?$

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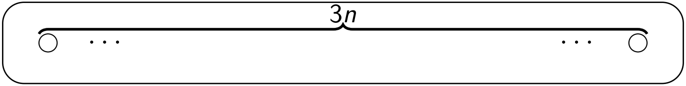

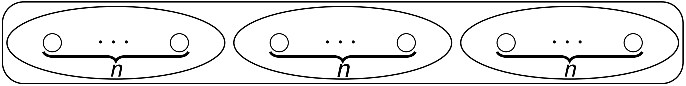

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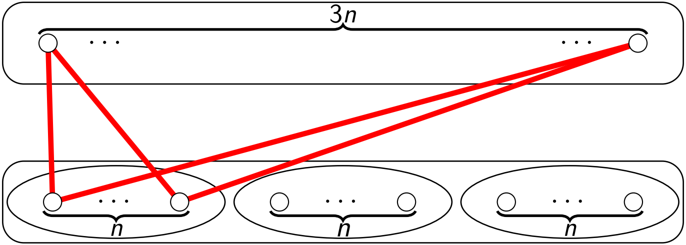

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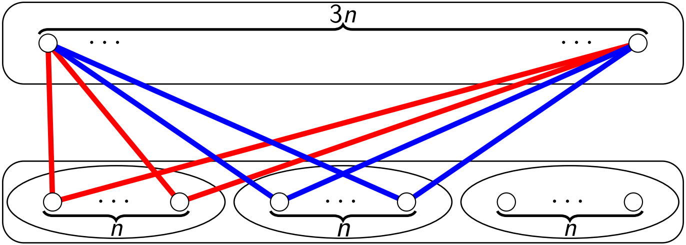

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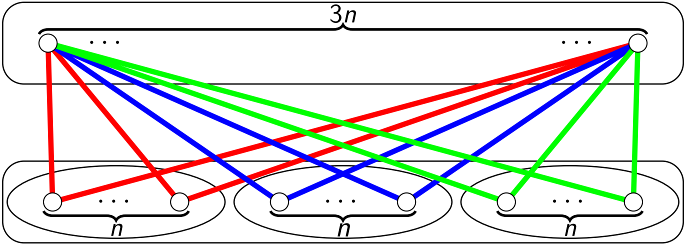

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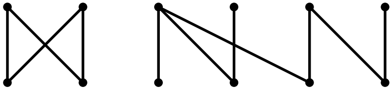

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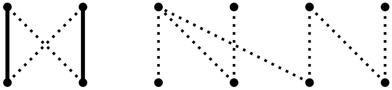

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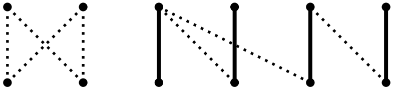

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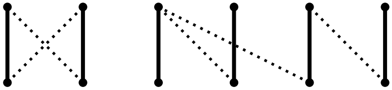

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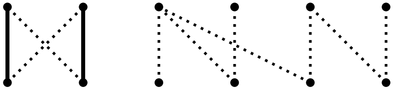

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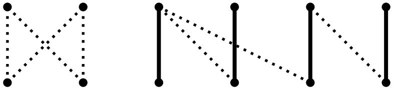

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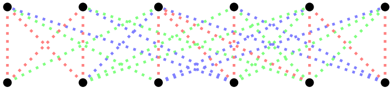

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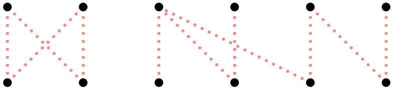

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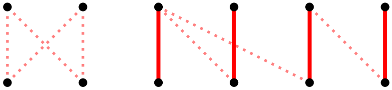

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- **Part 3:** Apply Part 2 to the 'reduced graph' obtained by the Szemeredi regularity lemma; then lift to find the desired cycle.

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- $G \setminus S$ is a 3-colouring of $K_{3n-2,3n-2}$ without monochromatic connected n-matchings, a contradiction to the inductive assumption.

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Theorem

Let $0 < \varepsilon < 2^{-20}$, *n large, and* $N \geq (3 + 2^{20}\varepsilon)n$. *Let G be a subgraph of* $K_{N,N}$ *of minimum degree at least* $(1 - \varepsilon)N$. *Then, if G is* 3*-coloured, there is a monochromatic connected n-matching.*

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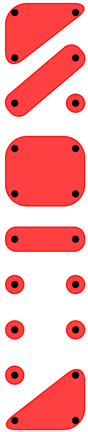

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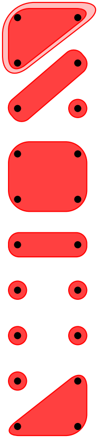

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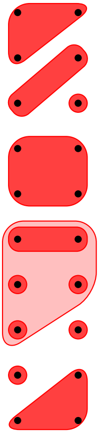

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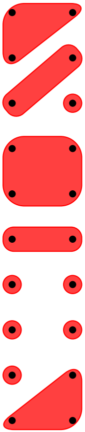

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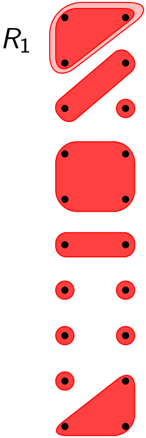

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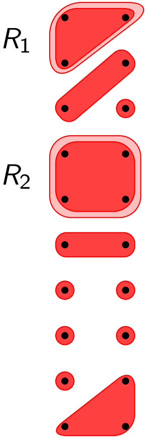

Part 2: almost complete bipartite graph

Theorem

Let $0 < \varepsilon < 2^{-20}$, n large, and $N \geq (3 + 2^{20}\varepsilon)n$. Let G be a subgraph of $K_{N,N}$ of minimum degree at least $(1 - \varepsilon)N$. Then, if G is 3-coloured, there is a monochromatic connected n-matching.

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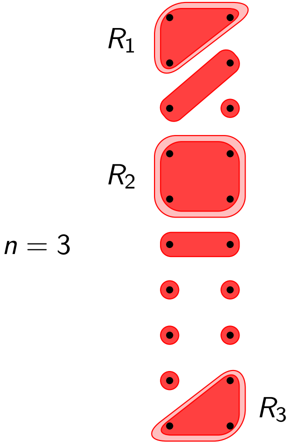

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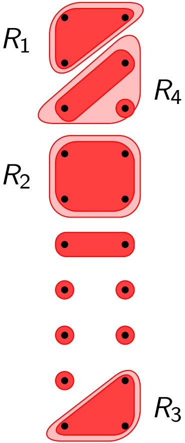

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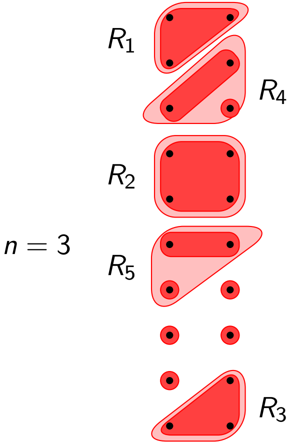

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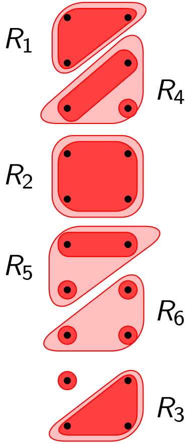

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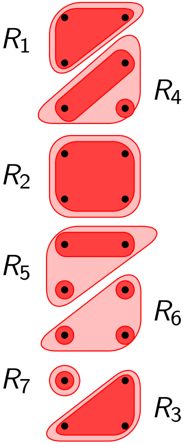

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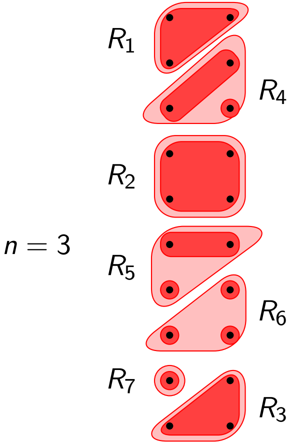

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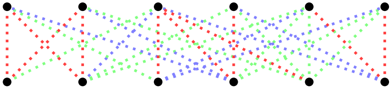

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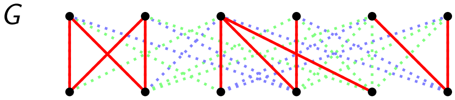

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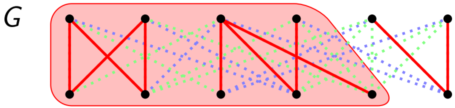

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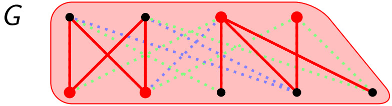

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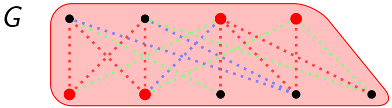

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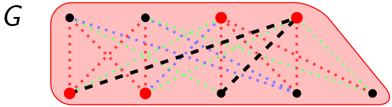

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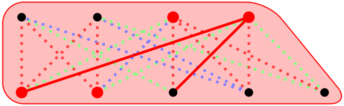

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- There are at most 8^6 types, so we are done.

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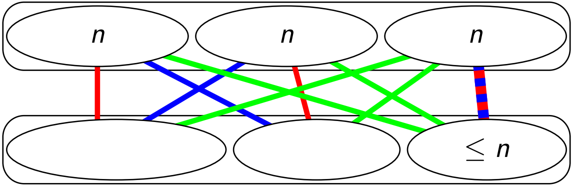

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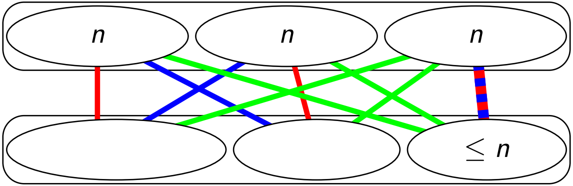

- Asymmetric results?

Thank You!