Covering random graphs by monochromatic trees and Helly-type results in hypergraphs

Matija Bucić
joint work with Daniel Korándi and Benny Sudakov

ETH Zürich

Covering by trees

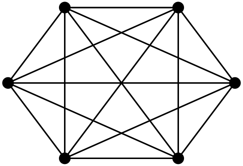
Definition

$\mathrm{tc}_r(G)$ *is the smallest* m *s.t. in any* r*-edge colouring of a graph* G *we can find* m *monochromatic trees which cover all vertices of* G.

Covering by trees

Definition

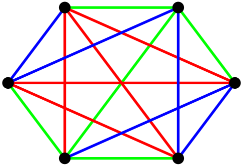
$\text{tc}_r(G)$ *is the smallest m s.t. in any r-edge colouring of a graph G we can find m monochromatic trees which cover all vertices of G.*



Covering by trees

Definition

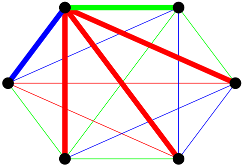
$\mathrm{tc}_r(G)$ *is the smallest* m *s.t. in any* r*-edge colouring of a graph* G *we can find* m *monochromatic trees which cover all vertices of* G.



Covering by trees

Definition

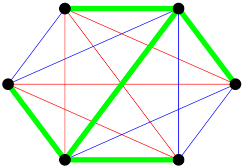
$\mathrm{tc}_r(G)$ *is the smallest* m *s.t. in any* r*-edge colouring of a graph* G *we can find* m *monochromatic trees which cover all vertices of* G.



Covering by trees

Definition

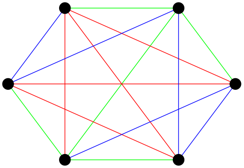
$\mathrm{tc}_r(G)$ *is the smallest* m *s.t. in any* r*-edge colouring of a graph* G *we can find* m *monochromatic trees which cover all vertices of* G*.*



Covering by trees

Definition

$\mathrm{tc}_r(G)$ is the smallest m s.t. in any r-edge colouring of a graph G we can find m monochromatic trees which cover all vertices of G.

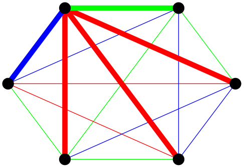


- $\mathrm{tc}_r(K_n) \leq r$

Covering by trees

Definition

$\text{tc}_r(G)$ *is the smallest* m *s.t. in any* r*-edge colouring of a graph* G *we can find* m *monochromatic trees which cover all vertices of* G.

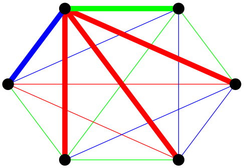


- $\text{tc}_r(K_n) \leq r$

Covering by trees

Definition

$\mathrm{tc}_r(G)$ is the smallest m s.t. in any r-edge colouring of a graph G we can find m monochromatic trees which cover all vertices of G.



- $\mathrm{tc}_r(K_n) \leq r$

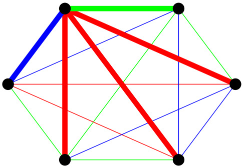
Conjecture (Lovasz '75, Ryser '70)

$$\mathrm{tc}_r(K_n) = r - 1$$

Covering by trees

Definition

$\text{tc}_r(G)$ is the smallest m s.t. in any r-edge colouring of a graph G we can find m monochromatic trees which cover all vertices of G.



- $\text{tc}_r(K_n) \le r$

Conjecture (Lovasz '75, Ryser '70)

$$\text{tc}_r(K_n) = r - 1$$

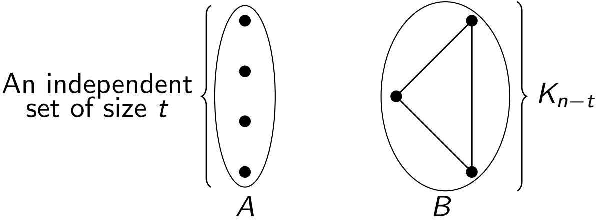
- For specific trees: Gyárfás; Gerencsér and Gyárfás; Pokrovskiy; Gyárfás, Ruszinkó, Sárközy and Szemerédi; Erdős, Gyárfás and Pyber.

Claim

There are graphs missing only a few edges with arbitrarily large tc_r .

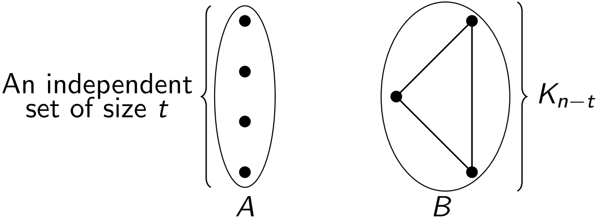
Claim

There are graphs missing only a few edges with arbitrarily large tc_r .



Claim

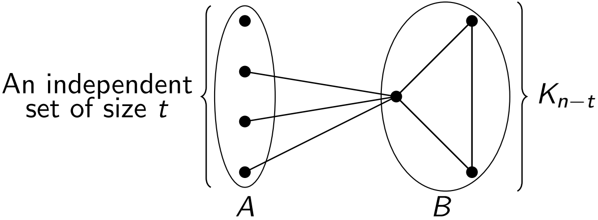
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.

Claim

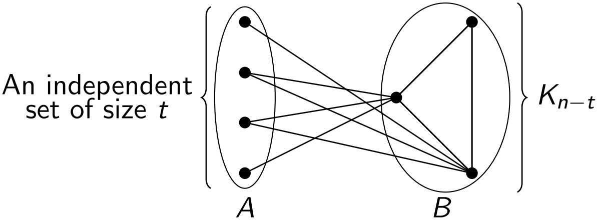
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r - 1$ edges to A.

Claim

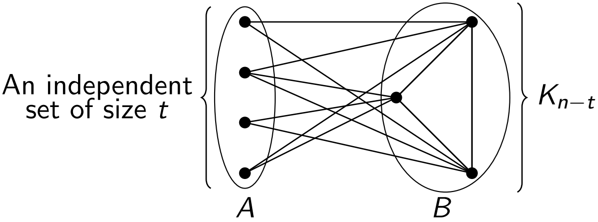
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.

Claim

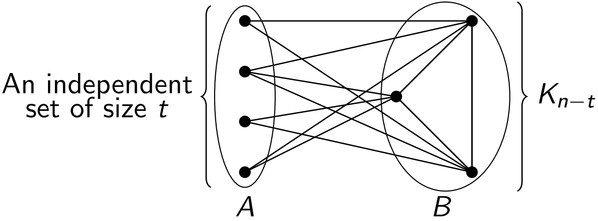
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.

Claim

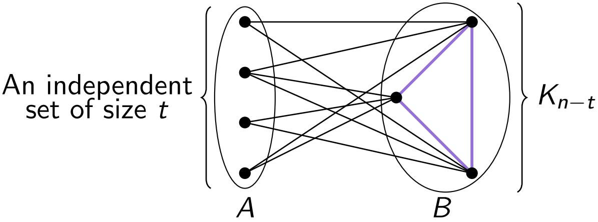
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.

Claim

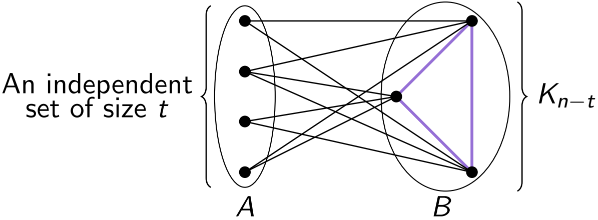
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.

Claim

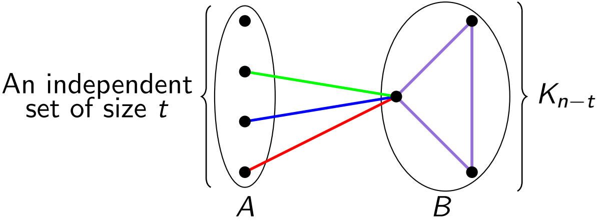
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.

Claim

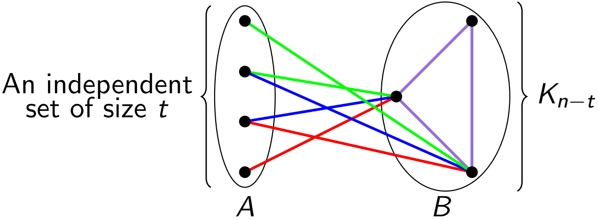
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.

Claim

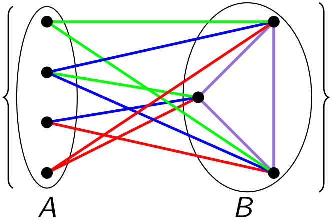
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.

Claim

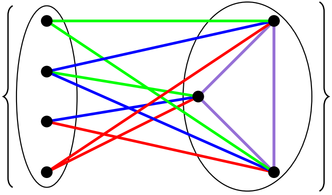
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.

Claim

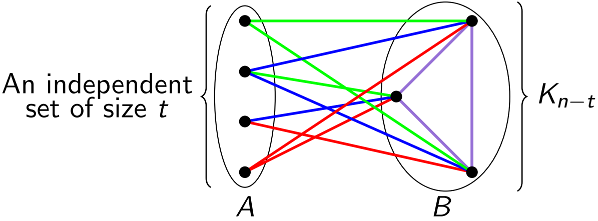
There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.
- Any monochromatic tree in first $r-1$ colours is a star with a centre in A.

Claim

There are graphs missing only a few edges with arbitrarily large tc_r .



- $\forall v \in B$ sends exactly $r-1$ edges to A.
- Colour all edges in B in colour r.
- $\forall v \in B$ colour its $r-1$ edges to A rainbowly using first $r-1$ colours.
- Any monochromatic tree in first $r-1$ colours is a star with a centre in A.

Proposition

If G contains an independent set of size t with no r vertices having a common neighbour then $\mathrm{tc}_r(G) \geq t$.

Covering by trees in random graphs

Bal and DeBiasio '15: What is $\text{tc}_r(\mathcal{G}(n,p))$?

Covering by trees in random graphs

Bal and DeBiasio '15: What is $\mathrm{tc}_r(\mathcal{G}(n,p))$?

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$

Covering by trees in random graphs

Bal and DeBiasio '15: What is $\text{tc}_r(\mathcal{G}(n,p))$?

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\text{tc}_r(\mathcal{G}(n,p)) \to \infty$

- np^r is roughly expected number of common neighbours of fixed r vertices

Covering by trees in random graphs

Bal and DeBiasio '15: What is $\text{tc}_r(\mathcal{G}(n,p))$?

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\text{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\text{tc}_r(\mathcal{G}(n,p)) \le r^2$*.*

- np^r is roughly expected number of common neighbours of fixed r vertices

Covering by trees in random graphs

Bal and DeBiasio '15: What is $\mathrm{tc}_r(\mathcal{G}(n,p))$?

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2$.

- np^r is roughly expected number of common neighbours of fixed r vertices
- If $\alpha(G) \geq r$ then $\mathrm{tc}_r(G) \geq r$.

Bal and DeBiasio '15: What is $\mathrm{tc}_r(\mathcal{G}(n,p))$?

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2$.

- np^r is roughly expected number of common neighbours of fixed r vertices
- If $\alpha(G) \geq r$ then $\mathrm{tc}_r(G) \geq r$.

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

- Proved for $r = 2$ by Kohayakawa, Mota and Schacht.

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

- Proved for $r = 2$ by Kohayakawa, Mota and Schacht.
- For $r \geq 3$ disproved by Ebsen, Mota and Schnitzer:

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

- Proved for $r = 2$ by Kohayakawa, Mota and Schacht.
- For $r \geq 3$ disproved by Ebsen, Mota and Schnitzer:

Example (Ebsen, Mota and Schnitzer)

For $r \geq 3$ *if* $p \ll \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \geq r + 1$.

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \le r$.

- Proved for $r = 2$ by Kohayakawa, Mota and Schacht.
- For $r \geq 3$ disproved by Ebsen, Mota and Schnitzer:

Example (Ebsen, Mota and Schnitzer)

For $r \geq 3$ *if* $p \ll \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \geq r+1$.

Theorem (B., Korándi, Sudakov)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{\sqrt{r}/2^{r-2}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) > r$

Conjecture (Bal and DeBiasio '15)

If $p \gg \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

- Proved for $r = 2$ by Kohayakawa, Mota and Schacht.
- For $r \geq 3$ disproved by Ebsen, Mota and Schnitzer:

Example (Ebsen, Mota and Schnitzer)

For $r \geq 3$ *if* $p \ll \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \geq r+1$.

Theorem (B., Korándi, Sudakov)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{\sqrt{r}/2^{r-2}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) > r$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/2^r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r$.

Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ then w.h.p. $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \le r^2$.

Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2$.

Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

- This establishes threshold for when $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$.

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \le r^2$.

Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

- This establishes threshold for when $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$. Established in weaker form by Korándi, Mousset, Nenadov, Škorić and Sudakov.

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2$.

Theorem (B., Korándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

- This establishes threshold for when $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$. Established in weaker form by Korándi, Mousset, Nenadov, Škorić and Sudakov.
- Answers a question of Lang and Lo.

Theorem (Bal and DeBiasio '15)

a) *If* $p \ll \left(\frac{\log n}{n}\right)^{1/r}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$ *and*

b) *If* $p \gg \left(\frac{\log n}{n}\right)^{1/(r+1)}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2$.

Theorem (B., Kordándi, Sudakov)

Let $d > 1$, $\left(\frac{\log n}{n}\right)^{\frac{1}{r}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{d(r+1)}}$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) = \Theta(r^2)$.

- This establishes threshold for when $\mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty$. Established in weaker form by Korándi, Mousset, Nenadov, Škorić and Sudakov.
- Answers a question of Lang and Lo.

Theorem (B., Korándi, Sudakov)

If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ *then w.h.p.* $\frac{r^2}{20 \log k} \leq \mathrm{tc}_r(\mathcal{G}(n,p)) \leq \frac{16 r^2 \log r}{\log k}$

Covering in hypergraphs

Question (Erdős, Hajnal and Tuza '90)

Given an r-uniform hypergraph H in which any k edges have a cover of size at most ℓ, how big can a cover of H be?

Covering in hypergraphs

Question (Erdős, Hajnal and Tuza '90)

Given an r-uniform hypergraph H in which any k edges have a cover of size at most ℓ, how big can a cover of H be?

- Studied by, Erdős, Fon-Der-Flaass, Kostochka and Tuza (1991); Fon-Der-Flaass, Kostochka and Woodall (1999); Kostochka (2001).

Covering in hypergraphs

Question (Erdős, Hajnal and Tuza '90)

Given an r-uniform hypergraph H in which any k edges have a cover of size at most ℓ, how big can a cover of H be?

- Studied by, Erdős, Fon-Der-Flaass, Kostochka and Tuza (1991); Fon-Der-Flaass, Kostochka and Woodall (1999); Kostochka (2001).
- A *transversal cover* in an r-partite r-uniform hypergraph H is a cover of H which has at most one vertex in each part of the r-partition.

Covering in hypergraphs

Question (Erdős, Hajnal and Tuza '90)

Given an r-uniform hypergraph H in which any k edges have a cover of size at most ℓ, how big can a cover of H be?

- Studied by, Erdős, Fon-Der-Flaass, Kostochka and Tuza (1991); Fon-Der-Flaass, Kostochka and Woodall (1999); Kostochka (2001).
- A *transversal cover* in an r-partite r-uniform hypergraph H is a cover of H which has at most one vertex in each part of the r-partition.

Definition

Let $\mathrm{hp}_r(k)$ *be the maximum possible size of a cover of an r-partite, r-uniform H in which any k edges have a transversal cover.*

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq \mathrm{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\mathrm{tc}_{r+1}(\mathcal{G}(n,p)) > \mathrm{hp}_r(k)$.

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq \mathrm{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\mathrm{tc}_{r+1}(\mathcal{G}(n,p)) > \mathrm{hp}_r(k)$.

- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\mathrm{tc}_r(\mathcal{G}(n,p)) \approx \mathrm{hp}_r(k)$.

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq \mathrm{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\mathrm{tc}_{r+1}(\mathcal{G}(n,p)) > \mathrm{hp}_r(k)$.

- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\mathrm{tc}_r(\mathcal{G}(n,p)) \approx \mathrm{hp}_r(k)$.

Conjecture (Bal and DeBiasio)

Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1 - \frac{1}{2^r})n$. Then vertices of G can be covered by monochromatic trees of distinct colours.

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\text{tc}_r(\mathcal{G}(n,p)) \leq \text{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\text{tc}_{r+1}(\mathcal{G}(n,p)) > \text{hp}_r(k)$.

- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\text{tc}_r(\mathcal{G}(n,p)) \approx \text{hp}_r(k)$.

Conjecture (Bal and DeBiasio)

Let G *be an* r*-coloured graph on* n *vertices with* $\delta(G) \geq (1 - \frac{1}{2^r})n$. *Then vertices of* G *can be covered by monochromatic trees of distinct colours.*

- Best possible in terms of $\delta(G)$.

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq \mathrm{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\mathrm{tc}_{r+1}(\mathcal{G}(n,p)) > \mathrm{hp}_r(k)$.

- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\mathrm{tc}_r(\mathcal{G}(n,p)) \approx \mathrm{hp}_r(k)$.

Conjecture (Bal and DeBiasio)

Let G *be an* r*-coloured graph on* n *vertices with* $\delta(G) \geq (1 - \frac{1}{2^r})n$*. Then vertices of* G *can be covered by monochromatic trees of distinct colours.*

- Best possible in terms of $\delta(G)$.
- Proved for $r \leq 3$ by Girão, Letzter and Sahasrabudhe.

The connection

Theorem (B., Korándi, Sudakov)

a) *Let* $k > r \geq 2$, $np^k \gg \log n$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq \mathrm{hp}_r(k)$.

b) *Let* $k > r \geq 2$, $np^{k+1} \ll \log n$ *then w.h.p.* $\mathrm{tc}_{r+1}(\mathcal{G}(n,p)) > \mathrm{hp}_r(k)$.

- If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ then w.h.p. $\mathrm{tc}_r(\mathcal{G}(n,p)) \approx \mathrm{hp}_r(k)$.

Theorem (B., Korándi, Sudakov)

Let G *be an* r*-coloured graph on* n *vertices with* $\delta(G) \geq (1 - \frac{1}{2^r})n$. *Then vertices of* G *can be covered by monochromatic trees of distinct colours.*

- Best possible in terms of $\delta(G)$.
- Proved for $r \leq 3$ by Girão, Letzter and Sahasrabudhe.

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:

□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.

□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.

□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.

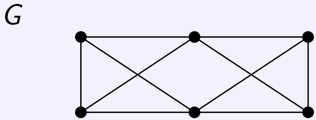
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



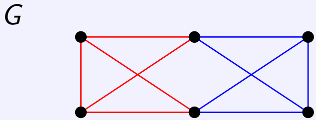
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



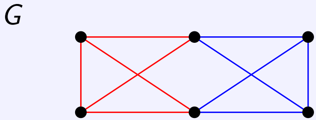
□

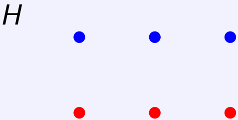
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





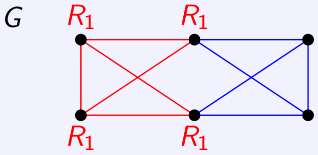
□

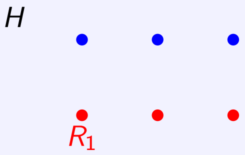
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





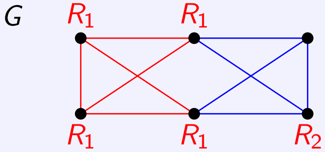
□

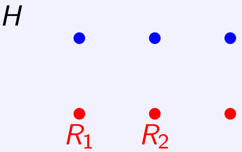
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





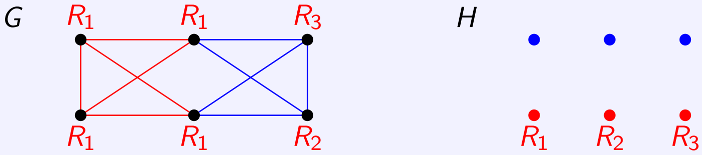
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \le \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



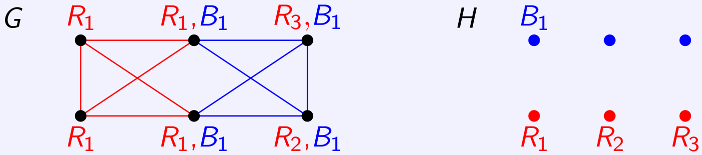
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



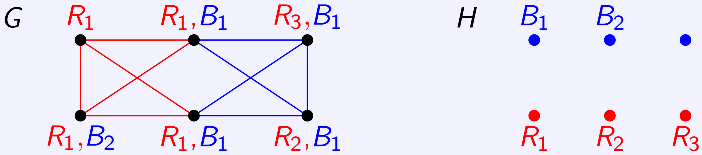
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\text{tc}_r(G) \leq \text{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



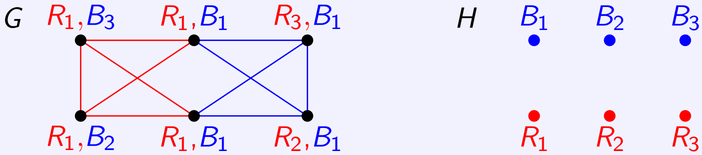
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



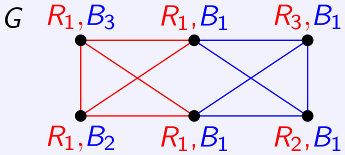
□

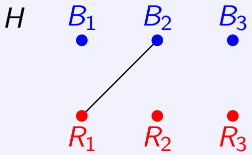
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





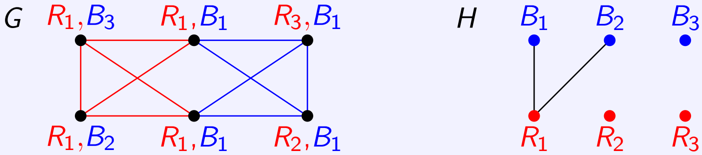
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



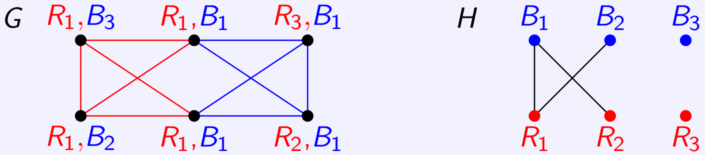
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



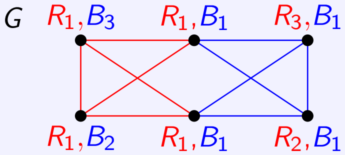
□

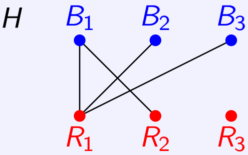
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





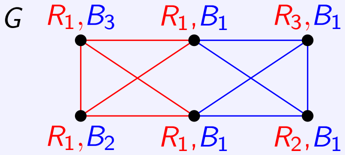
□

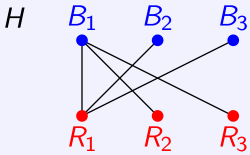
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\text{tc}_r(G) \leq \text{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





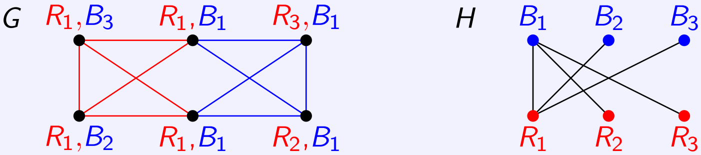
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



- Parts correspond to monochromatic components of the same colour.

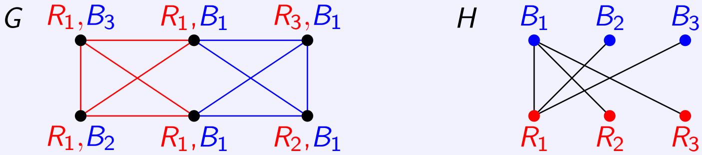
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



- Parts correspond to monochromatic components of the same colour.
- Given k edges $e_{v_1}, \ldots, e_{v_k}$, let w be a common neighbour of $v_1, \ldots, v_k$.

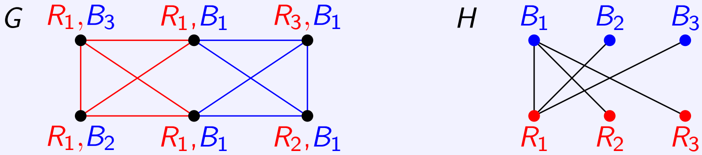
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\text{tc}_r(G) \leq \text{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



- Parts correspond to monochromatic components of the same colour.
- Given k edges $e_{v_1}, \ldots, e_{v_k}$, let w be a common neighbour of $v_1, \ldots, v_k$.
- $e_w \cap e_{v_i} \neq \emptyset$ so e_w is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq \text{hp}_r(k)$.

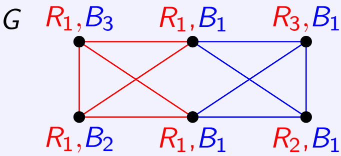
□

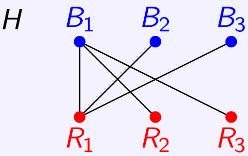
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \le \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





- Parts correspond to monochromatic components of the same colour.
- Given k edges $e_{v_1}, \dots, e_{v_k}$, let w be a common neighbour of $v_1, \dots, v_k$.
- $e_w \cap e_{v_i} \neq \emptyset$ so e_w is a transversal cover of $e_{v_1}, \dots, e_{v_k}$ so $\tau(H) \le \mathrm{hp}_r(k)$.
- If monochromatic components $C_1, \dots C_t$ cover H then $C_1 \cup \cdots \cup C_t = V(G)$

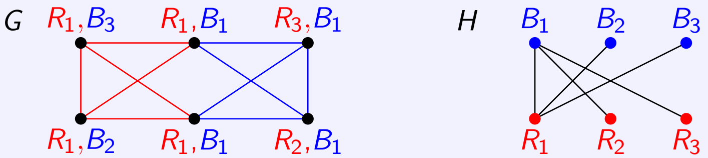
□

Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.



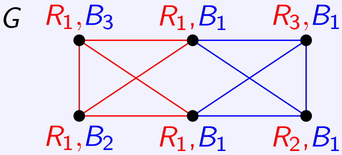
- Parts correspond to monochromatic components of the same colour.
- Given k edges $e_{v_1}, \ldots, e_{v_k}$, let w be a common neighbour of $v_1, \ldots, v_k$.
- $e_w \cap e_{v_i} \neq \emptyset$ so e_w is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq \mathrm{hp}_r(k)$.
- If monochromatic components $C_1, \ldots C_t$ cover H then $C_1 \cup \cdots \cup C_t = V(G)$
  - For any $v \in V(G)$ $\exists i : C_i \in e_v$ □

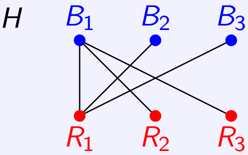
Theorem (B., Korándi, Sudakov)

If any k vertices in G have a common neighbour then $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

Proof.

- Given r-colouring of G we build r-partite r-uniform hypergraph H:
  - Vertices of H are monochromatic components of G.
  - Let e_v be the set of monochromatic components of G containing v.
  - $E(H) := \{e_v \mid v \in V(G)\}$.





- Parts correspond to monochromatic components of the same colour.
- Given k edges $e_{v_1}, \ldots, e_{v_k}$, let w be a common neighbour of $v_1, \ldots, v_k$.
- $e_w \cap e_{v_i} \neq \emptyset$ so e_w is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq \mathrm{hp}_r(k)$.
- If monochromatic components $C_1, \ldots C_t$ cover H then $C_1 \cup \cdots \cup C_t = V(G)$
  - For any $v \in V(G)$ $\exists i : C_i \in e_v \implies v \in C_i$. □

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

$\square$

Theorem (B., Korándi, Sudakov)

$\text{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j.$

□

Theorem (B., Korándi, Sudakov)

$\text{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since B_j is a cover of $H - A_j$.

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since B_j is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \dots, V_r$ be parts and $A_1, \dots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since B_j is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$ since otherwise B_i is a transversal cover of H, a contradiction.

□

Theorem (B., Korándi, Sudakov)

$\text{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \dots, V_r$ be parts and $A_1, \dots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since B_j is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$ since otherwise B_i is a transversal cover of H, a contradiction.
- This implies $|E(H)| = k \leq \prod_{i=1}^{r} \binom{1+1}{1} = 2^r$,

□

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$.

Proof.

- AFSOC there is an r-uniform, r-partite hypergraph without a transversal cover but with any 2^r of its edges having a transversal cover.
- Let H be minimal satisfying above conditions.
- Let $V_1, \ldots, V_r$ be parts and $A_1, \ldots, A_k$ edges of H.
- Let B_i be a transversal cover of $H - A_i$. Then:
  - $|A_i \cap V_j| = |B_i \cap V_j| = 1, \forall i, j$.
  - $A_i \cap B_j \neq \emptyset, \forall i \neq j$ since B_j is a cover of $H - A_j$.
  - $A_i \cap B_i = \emptyset, \forall i$ since otherwise B_i is a transversal cover of H, a contradiction.
- This implies $|E(H)| = k \leq \prod_{i=1}^{r} \binom{1+1}{1} = 2^r$,
- So H has a transversal cover, a contradiction ↯. □

Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$ *and* $\mathrm{hp}_r(2^r/\sqrt{r}) > r$.

Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$ *and* $\mathrm{hp}_r(2^r/\sqrt{r}) > r$.

Question

Is the $\sqrt{r}$ *necessary?*

Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$ *and* $\mathrm{hp}_r(2^r/\sqrt{r}) > r$.

Question

Is the $\sqrt{r}$ *necessary?*

Theorem (B., Korándi, Sudakov)

$\frac{r^2}{12\log k} \leq \mathrm{hp}_r(k) \leq \frac{16r^2 \log r}{\log k}$.

Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$ *and* $\mathrm{hp}_r(2^r/\sqrt{r}) > r$.

Question

Is the $\sqrt{r}$ *necessary?*

Theorem (B., Korándi, Sudakov)

$\frac{r^2}{12\log k} \leq \mathrm{hp}_r(k) \leq \frac{16r^2 \log r}{\log k}$.

Question

Is the $\log r$ *necessary?*

Concluding remarks and open problems

Theorem (B., Korándi, Sudakov)

$\mathrm{hp}_r(2^r) = r$ *and* $\mathrm{hp}_r(2^r/\sqrt{r}) > r$.

Question

Is the $\sqrt{r}$ *necessary?*

Theorem (B., Korándi, Sudakov)

$\frac{r^2}{12 \log k} \leq \mathrm{hp}_r(k) \leq \frac{16r^2 \log r}{\log k}$.

Question

Is the $\log r$ *necessary? Same question without* r*-partiteness condition.*


рахмат
danke
謝謝
ngiyabonga
spas
teşekkür ederim
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
kiitos
dankie
nanni
nandri
dhanyavad
thank you
gracias
хвала
asante
manana
obrigada
tenki
bedankt
bayarlalaa
gràcie
hvala
maururu
köszönöm
akun dankon
aciū
djiere dieuf
tau
mochchakkeram
murakoze
enkosi
дякую
mamnun
dziękuję
chnorakaloutioun
gratias ago
gràcies
sulpáy
go raibh maith agat
chokrane
sobodi
dėkuji
sukriya
kop khun krap
taiku
mèsi
didi madloba
sagolun
ありがとう
tanemirt
grazie
arigatô
takk
dakujem
trugarez
najis tuke
terima kasih
rahmet
diolch
dhanyavadagalu
shukriya
mercê
мерси
kam sah hamnida
rahmat
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci
תודה רבה