The intersection spectrum of 3-chromatic intersecting hypergraphs

Matija Bucić
joint work with Stefan Glock and Benny Sudakov

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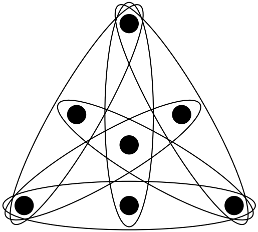

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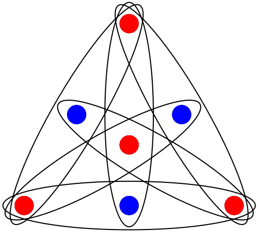

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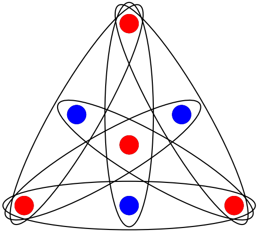

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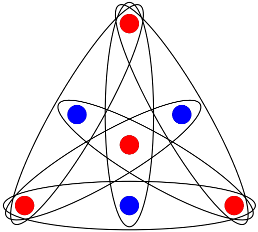

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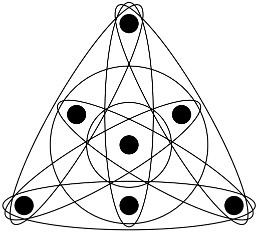

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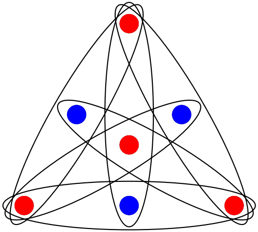

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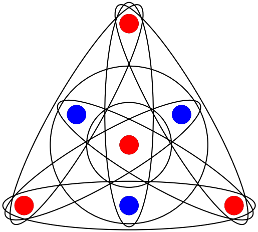

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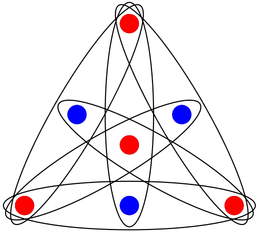

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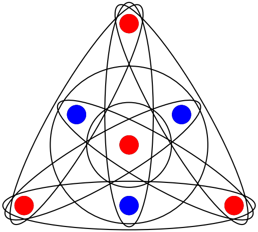

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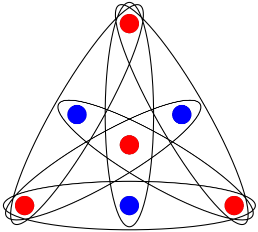

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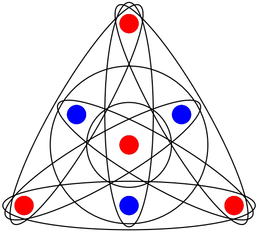

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- State of the art:

$$k^{1/2-o(1)} \ll m(k)/2^k \ll k^2$$

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Show $\widetilde{m}(k) = (2+o(1))^k$.

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- What about non-2-colorable intersecting k-graphs?

Problem 2: maximum intersection size.

Theorem (Erdős and Lovász; Shelah)

For any non-2-colorable intersecting k-graph H we have $\max I(H) \geq \frac{k}{\log k}$.

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- After $\frac{k}{\log k}$ many steps we still have $2^k/k^{k/\log k} > 1$ edges. □

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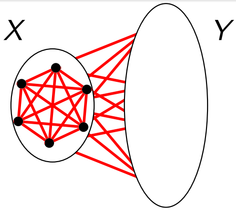

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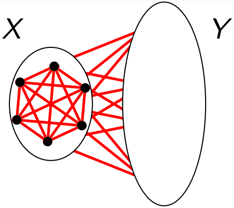

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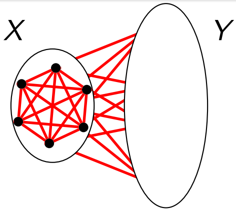

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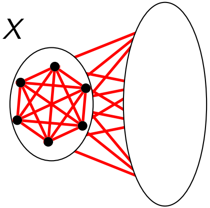

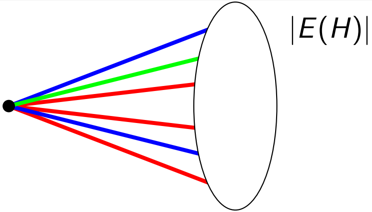

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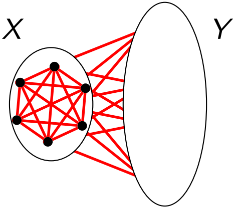

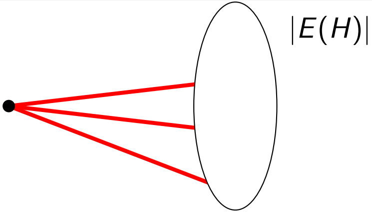

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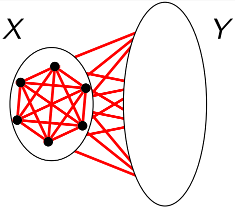

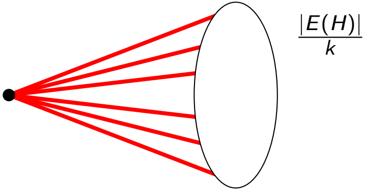

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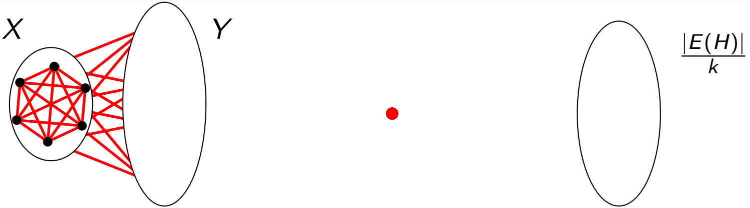

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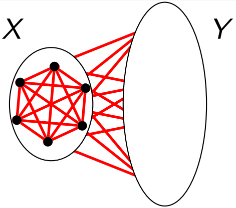

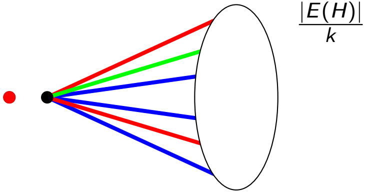

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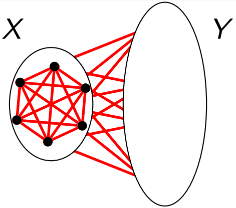

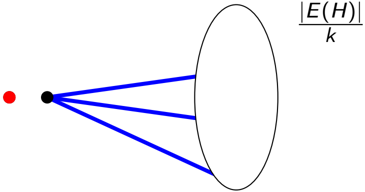

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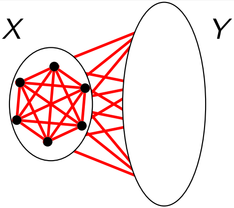

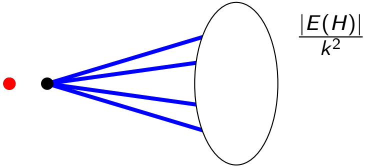

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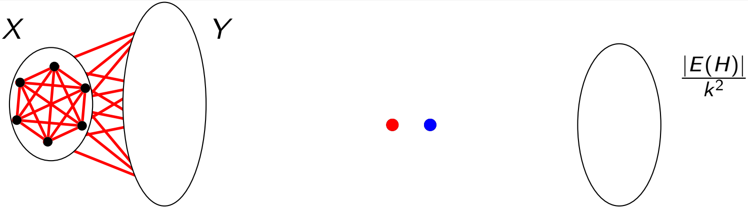

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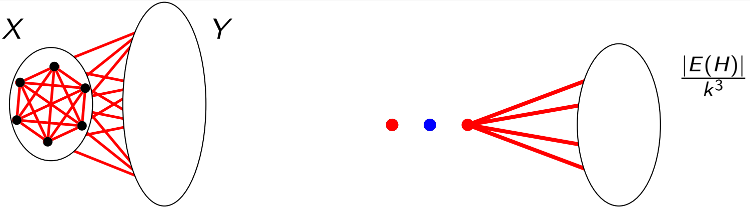

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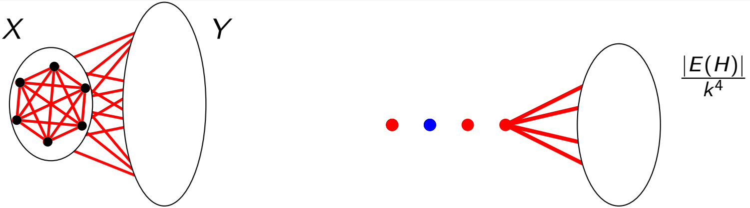

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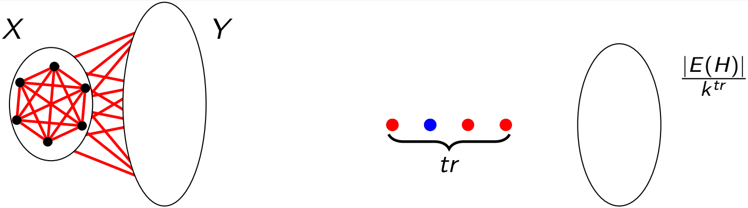

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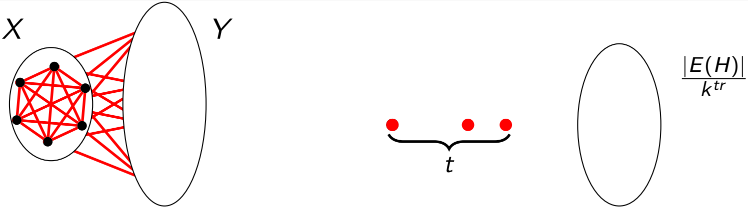

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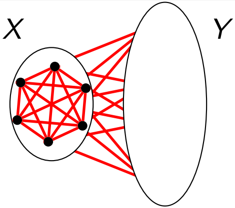

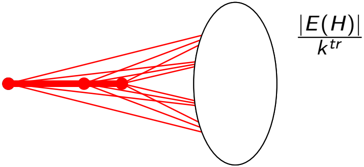

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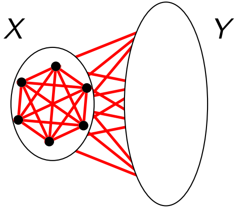

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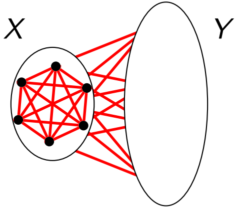

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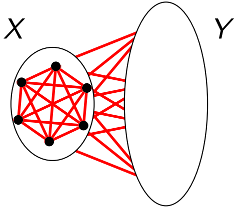

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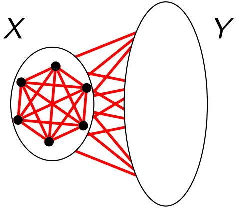

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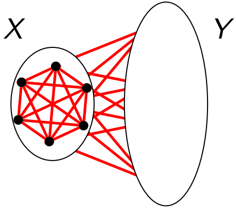

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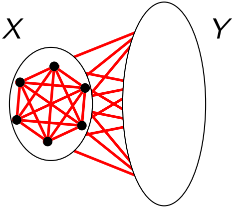

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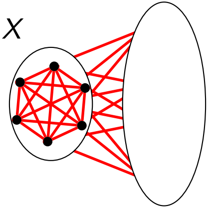

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Proof.

- Let s_x, t_x denote the number of edges in S, T containing x, respectively.

□

Density increment

- Let $S, T \subseteq E(H)$ be disjoint. We set
  $\lambda_S :=$ average intersection size inside S and
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- Let s_x, t_x denote the number of edges in S, T containing x, respectively.
- $\binom{s_x}{2} + \binom{t_x}{2} = \frac{s_x^2+t_x^2}{2} - \frac{s_x+t_x}{2} \geq s_x t_x - \frac{s_x+t_x}{2}$. □

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Let $S, T \subseteq E(H)$ be disjoint and of size ℓ.

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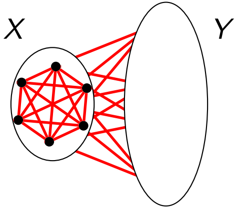
Average intersections in Y are large

Goal: given a λ_i-pair (X, Y) with $|X| = k^{1/3}$ and $|Y|$ big for some $j > i$ find a λ_j-pair (X', Y') with $|X'| = k^{1/3}$ and $|Y'| \geq |Y|/k^{O(k^{2/3})}$.



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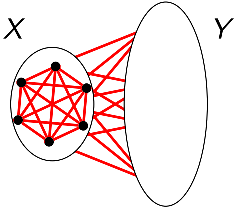

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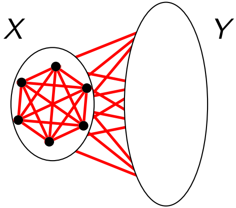

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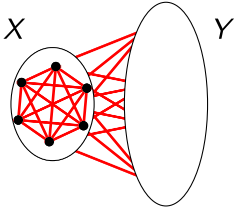

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- Implies it for bigger R as well by averaging. □

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Goal: given a λ_i-pair (X, Y) with $|X| = k^{1/3}$ and $|Y|$ big for some $j > i$ find a λ_j-pair (X', Y') with $|X'| = k^{1/3}$ and $|Y'| \geq |Y|/k^{O(k^{2/3})}$.

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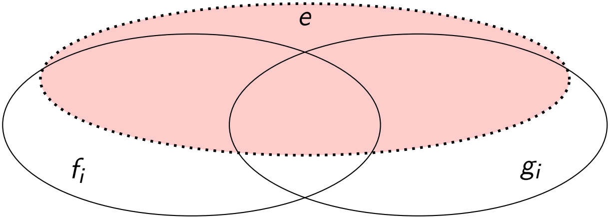

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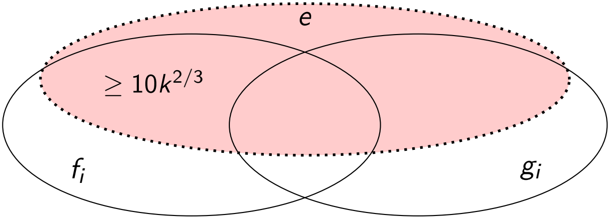

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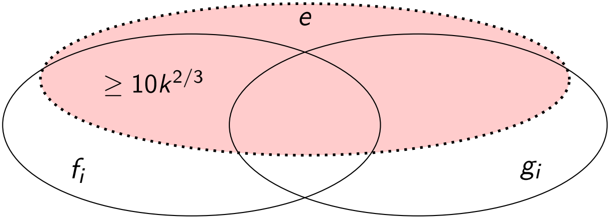

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- Any $f, g \in Y' \subseteq Y$ differ by less than $10k^{2/3}$ within e.

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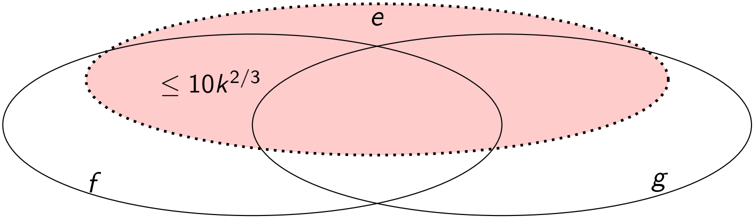

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- $|Y'| \geq |Y|/2$.
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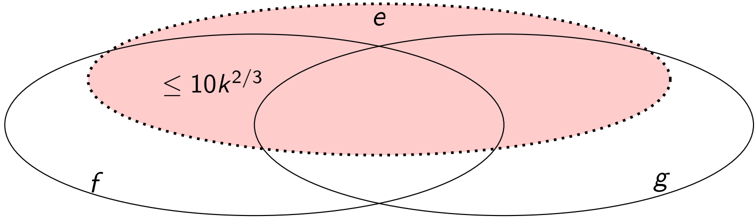

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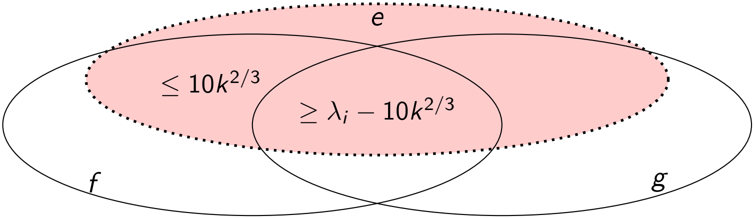

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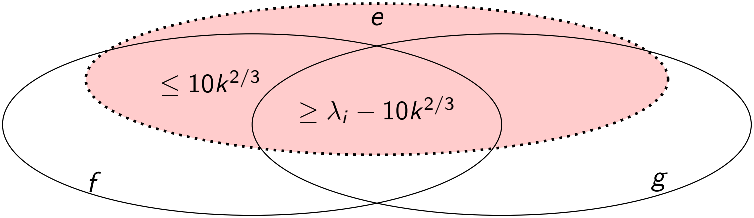

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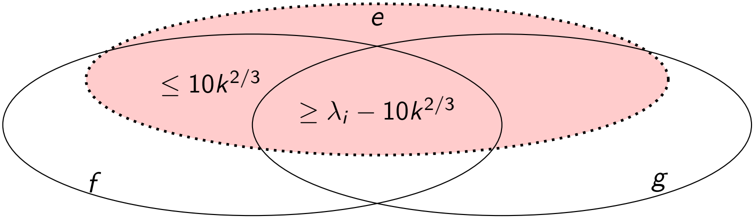
- Pay $k^{10k^{2/3}+1}$ to find a set of $> \lambda_i$ vertices contained in $\geq \frac{|Y|}{k^{O(k^{2/3})}}$ edges.



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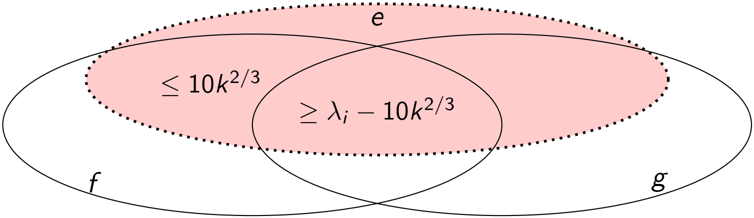
- Use Ramsey lemma again within this set to find a new pair.



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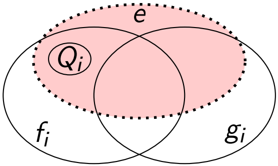

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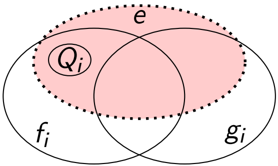
- $m \geq |Y|/4$, let $Q_i \subseteq (f_i \setminus g_i) \cap e$, of size $|Q_i| = 10k^{2/3}$
- There are at most $\binom{|e|}{10k^{2/3}}$ different choices for Q_i.



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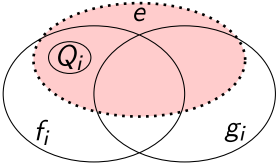

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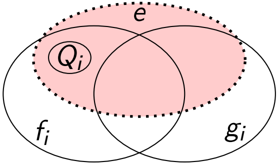

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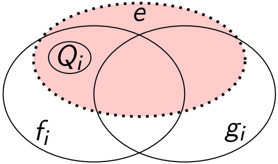

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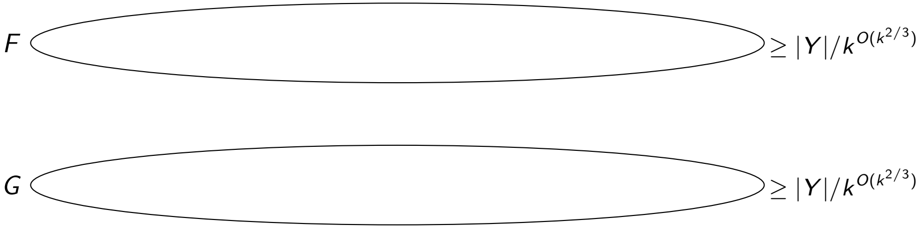

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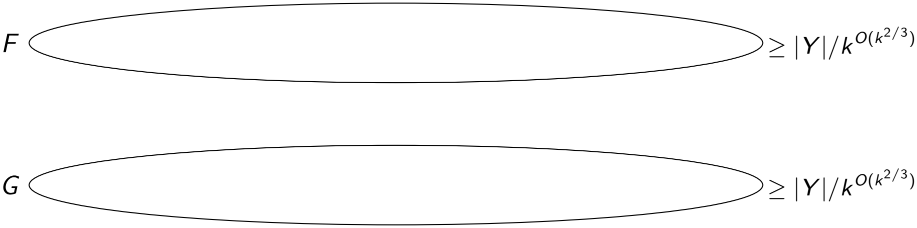

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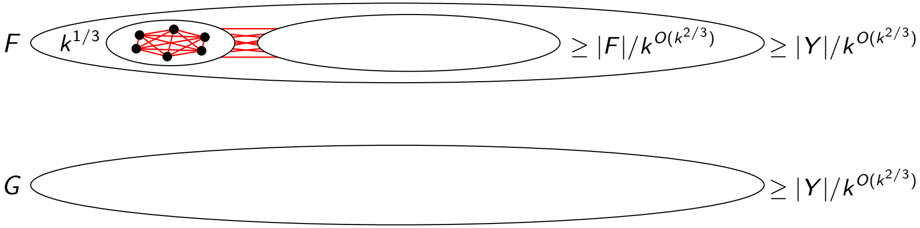

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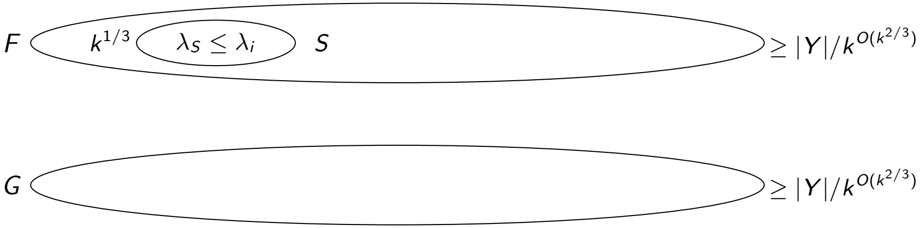

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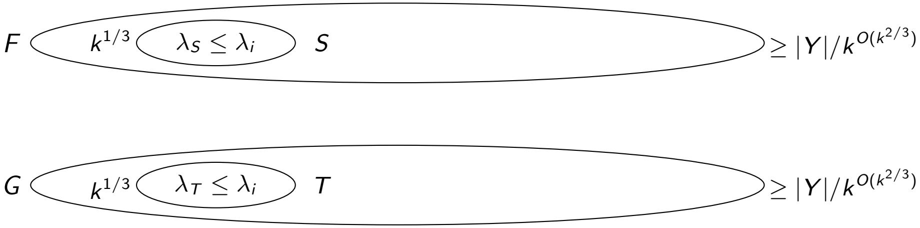

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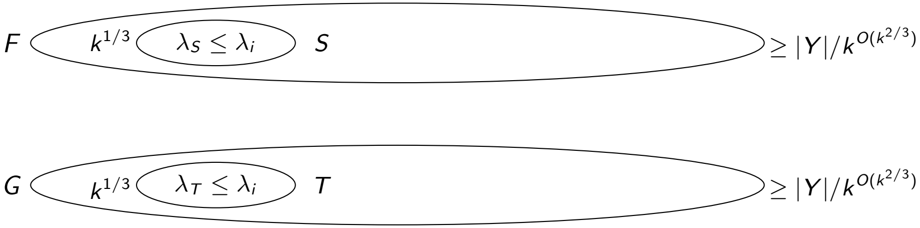

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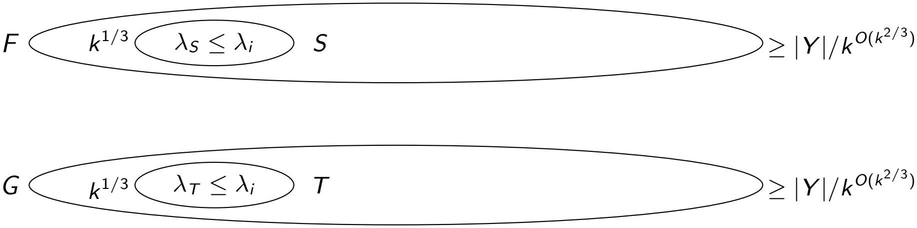

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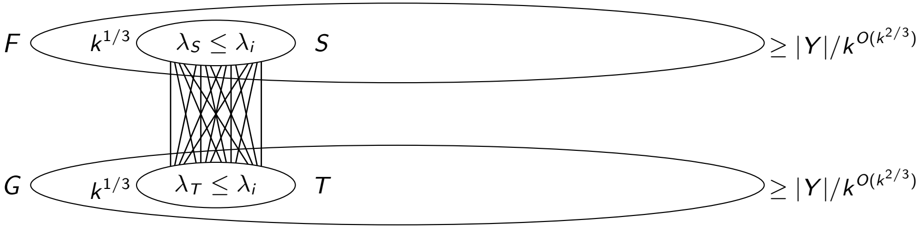

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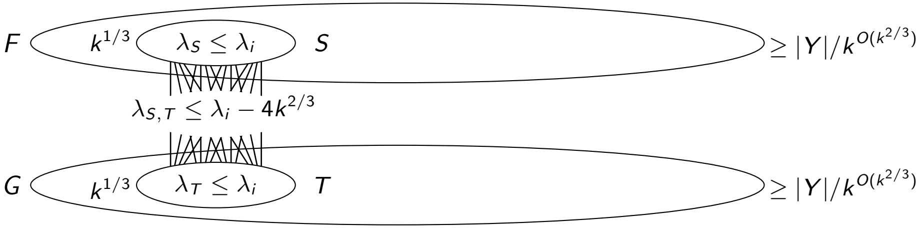

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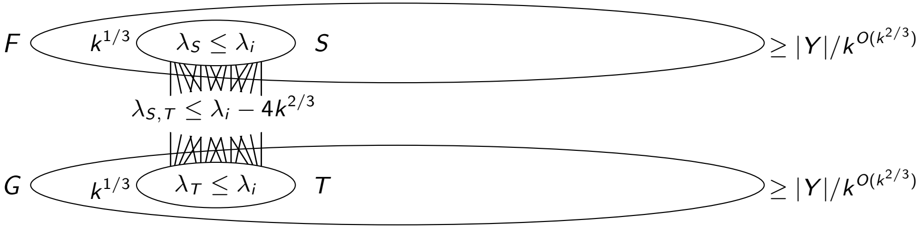
- Use stronger corollary to deduce $\lambda_{S,T} \leq \lambda_i - 4k^{2/3}$.
- This gives $\lambda_{S \cup T} < \lambda_i - 2k^{2/3}$, which is a contradiction.



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Problem 1

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рахмат
danke
謝謝
ngiyabonga
spas
teşekkür ederim
Баярлалаа
mersi
kia ora
barka
welalin
tack
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
kiitos
dankie
thank you
gracias
хвала
nanni
nandri
dhanyavad
mauruuru
köszönöm
aciu
asante
manana
bedankt
bayarlalaa
gràcie
hvala
akun
dankon
obrigada
murakoze
tenki
enkosi
dziękuję
djiere dieuf
tau
mochchakkeram
дякую
mamnun
chokrane
chnorakaloutioun
gratias ago
gràcies
sulpáy
go raibh maith agat
sobodi
dėkuji
sagolun
sukriya
kop khun krap
taiku
obrigado
mèsi
didi madloba
ありがとう
tanemirt
grazie
arigatô
takk
dakujem
trugarez
najis tuke
terima kasih
rahmet
diolch
dhanyavadagalu
shukriya
mercè
мерси
kam sah hamnida
rahmat
xièxie
ευχαριστώ
তোমাকে ধন্যবাদ
감사합니다
merci
תודה רבה