Appendix A. The relevant projections and their properties.

The definition and computation of the singular matrix $\widehat{\mathbf{A}} := \mathbf{PAQ}$, which replaces $\mathbf{A}$ in the Krylov solver restricted to $\mathcal{R}(\mathbf{P})$, is based on suitably chosen projections $\mathbf{P}$ and $\mathbf{Q}$. For deflated CG [41, 13, 46] it is well known how to choose $\mathbf{P}$ and $\mathbf{Q}$; see Nicolaides [41], Dostál [13], our companion paper for Hermitian problems [27], and recent work where the usage of such projections in deflation, domain decomposition, and multigrid has been pointed out [18, 40, 53].

Here, we procure a family of projections that is suitable for deflating techniques for non-Hermitian systems, which, in particular, can also be applied to BiCG. We present lemmas and theorems with lists of properties, including some intermediate results used to prove the more fundamental ones; however, we refrain from giving all the details of proof. Certain members of the family were proposed in the literature already, specifically by Abdel-Rehim, Morgan, and Wilcox [1] for their deflated BiCGStab, by Abdel-Rehim, Stathopoulos, and Orginos [2] for their Lanczos based combined equation and eigenvalue solver, by Ahuja [5] in his proposal for a deflated version of BiCG, and by Erlangga and Nabben [16] for their version of deflated GMRes. However, in none of these cases the resulting deflated Krylov subspace method was mathematically identical to one of our new proposals.

While in Table 2.1 of the paper we just list without proof the most relevant properties of the projectors and the resulting operators $\widehat{\mathbf{A}}$ and $\widetilde{\mathbf{A}}$, in this appendix here we collect many more results in a way that makes it easy to verify them.

As before, let $\mathbf{A} \in \mathbb{C}^{N\times N}$ be nonsingular, and let $\mathbf{U}, \widetilde{\mathbf{U}} \in \mathbb{C}^{N\times k}$ (with $1 \le k < N$) have full rank, so that its columns define a basis of a projection subspace and so do the columns of

$$\mathbf{Z} := \mathbf{AU}\,, \qquad \widetilde{\mathbf{Z}} := \mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{U}}\,.$$

We assume that $\widetilde{\mathbf{U}}^{\mathsf{H}}\mathbf{BAU} \in \mathbb{C}^{k\times k}$ and $\mathbf{B} \in \mathbb{C}^{N\times N}$ are nonsingular. Later, when deflating non-Hermitian Lanczos-type methods, we additionally assume that $\mathbf{B}$ commutes with $\mathbf{A}$ and $\mathbf{P}$. We are mainly interested in the cases $\mathbf{B} = \mathbf{I}$ and $\mathbf{B} = \mathbf{A}$. But we also want to include the case $\mathbf{B} = \mathbf{A}^{\mathsf{H}}$, which is the right choice for GMRes. We further let

$$\mathbf{E} := \mathbf{E}_{\mathbf{B}} := \widetilde{\mathbf{U}}^{\mathsf{H}}\mathbf{BAU}\,, \qquad \mathbf{M} := \mathbf{M}_{\mathbf{B}} := \mathbf{U}\mathbf{E}_{\mathbf{B}}^{-1}\widetilde{\mathbf{U}}^{\mathsf{H}}\,. \tag{A.1}$$

For simplicity we also define

$$\mathcal{U} := \mathcal{R}(\mathbf{U})\,, \qquad \widetilde{\mathcal{U}} := \mathcal{R}(\widetilde{\mathbf{U}})\,, \qquad \mathcal{Z} := \mathcal{R}(\mathbf{Z})\,, \qquad \widetilde{\mathcal{Z}} := \mathcal{R}(\widetilde{\mathbf{Z}})\,.$$

Lemma A.1. $\mathsf{rank}\,\mathbf{M} = k$.

Proof. Because $\mathbf{U}$ has full rank and $\mathbf{E}$ is nonsingular, we obtain $\mathcal{N}(\mathbf{M}) = \mathcal{N}(\widetilde{\mathbf{U}}^{\mathsf{H}}) = \mathcal{R}(\widetilde{\mathbf{U}})^{\perp}$. With $\mathsf{rank}\,\widetilde{\mathbf{U}} = k$ we have $\dim \mathcal{N}(\mathbf{M}) = N - k$, and the statement follows from the rank-nullity theorem. ☐

$\mathbf{M}$ has the important feature that $\mathbf{AMB}$ and $\mathbf{MBA}$ are both oblique projections. This and related results are gathered next.

Lemma A.2. *The following properties of* $\mathbf{M}$ *hold:*

1) $\mathbf{MBAU} = \mathbf{U}$ *;*
2) $\mathbf{MBAM} = \mathbf{M}$ *;*
3) $\mathcal{R}(\mathbf{M}) = \mathcal{U}$ *;* $\mathcal{N}(\mathbf{M}) = \widetilde{\mathcal{U}}^{\perp}$ *;*
4) $\mathcal{R}(\mathbf{M}^{\mathsf{H}}) = \widetilde{\mathcal{U}}$ *;* $\mathcal{N}(\mathbf{M}^{\mathsf{H}}) = \mathcal{U}^{\perp}$ *;*
5) $\mathcal{N}(\mathbf{AMB}) = \mathcal{N}(\mathbf{MB}) = [\mathcal{R}(\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp} = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *;*

6) $\mathcal{N}(\mathbf{MBA}) = [\mathcal{R}(\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp}$ *;*

7) $\mathbf{AMB}$ *is the projection onto* $\mathcal{R}(\mathbf{AM}) = \mathcal{Z}$ *along* $\mathcal{N}(\mathbf{MB}) = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *;*

8) $\mathbf{MBA}$ *is the projection onto* $\mathcal{R}(\mathbf{M}) = \mathcal{U}$ *along* $\mathcal{N}(\mathbf{MBA}) = [\mathcal{R}(\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp}$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statements 6 and 8 can be reformulated as*

6') $\mathcal{N}(\mathbf{MBA}) = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{Z}})^{\perp}$ *;*

8') $\mathbf{MBA}$ *is the projection onto* $\mathcal{R}(\mathbf{M}) = \mathcal{U}$ *along* $\mathcal{N}(\mathbf{MBA}) = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{Z}})^{\perp}$ *.*

Proof. If proven in the given order, the statements follow easily from the definition of $\mathbf{M}$ and the fact that $\mathsf{rank}\,\mathbf{M} = \mathsf{rank}\,\mathbf{U} = \mathsf{rank}\,\widetilde{\mathbf{U}} = k$. Of course, for several matrices $\mathbf{G}$, we make repeatedly use of the equality $\mathcal{N}(\mathbf{G}) = [\mathcal{R}(\mathbf{G}^{\mathsf{H}})]^{\perp}$. □

Next, we let

$$\mathbf{P} := \mathbf{I} - \mathbf{AMB}\,, \qquad \mathbf{Q} := \mathbf{I} - \mathbf{MBA}\,. \tag{A.2}$$

Then the above Lemma A.2 implies a series of properties of $\mathbf{P}$ and $\mathbf{Q}$, which, again, are easily verified.[8]

LEMMA A.3. *The following properties of* $\mathbf{P}$ *and* $\mathbf{Q}$ *hold:*

9) $\mathbf{PA} = \mathbf{AQ} = \mathbf{PAQ}$ *;*

10) $\mathsf{rank}\,\mathbf{P} = \mathsf{rank}\,\mathbf{Q} = N - k$ *;*

11) $\mathbf{QU} = \mathbf{O}$ *and* $\mathbf{QM} = \mathbf{O}$ *;*

12) $\mathbf{PZ} = \mathbf{PAU} = \mathbf{O}$ *and* $\mathbf{PAM} = \mathbf{O}$ *;*

13) $\widetilde{\mathbf{U}}^{\mathsf{H}}\mathbf{BP} = \mathbf{O}$ *and* $\mathbf{MBP} = \mathbf{O}$ *;*

14) $\widetilde{\mathbf{U}}^{\mathsf{H}}\mathbf{BAQ} = \mathbf{O}$ *and* $\mathbf{MBAQ} = \mathbf{O}$ *;*

15) $\mathcal{N}(\mathbf{P}) = \mathcal{Z} = \mathcal{R}(\mathbf{AM})$ *;*

16) $\mathcal{N}(\mathbf{Q}) = \mathcal{U} = \mathcal{R}(\mathbf{M})$ *;*

17) $\mathbf{P}$ *is the projection onto* $\mathcal{R}(\mathbf{P}) = \mathcal{N}(\mathbf{MB}) = [\mathcal{R}(\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp} = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *along* $\mathcal{N}(\mathbf{P}) = \mathcal{Z}$ *;*

18) $\mathbf{Q}$ *is the projection onto* $\mathcal{R}(\mathbf{Q}) = \mathcal{N}(\mathbf{MBA}) = [\mathcal{R}(\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp} = (\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *along* $\mathcal{N}(\mathbf{Q}) = \mathcal{U}$ *;*

19) $\mathcal{R}(\mathbf{PA}) = \mathcal{R}(\mathbf{P}) = \mathcal{N}(\mathbf{MB}) = [\mathcal{R}(\mathbf{B}^{\mathsf{H}}\mathbf{M}^{\mathsf{H}})]^{\perp} = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *;*

20) $\mathcal{N}(\mathbf{PA}) = \mathcal{N}(\mathbf{Q}) = \mathcal{U} = \mathcal{R}(\mathbf{M})$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statement 18 can be reformulated as*

18') $\mathbf{Q}$ *is the projection onto* $\mathcal{R}(\mathbf{Q}) = \mathcal{N}(\mathbf{MAB}) = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{Z}})^{\perp}$ *along* $\mathcal{N}(\mathbf{Q}) = \mathcal{U}$ *.*

In particular, for $\widehat{\mathbf{A}} := \mathbf{PA}$ the following theorem holds.

THEOREM A.4. *For the restrictions of* $\mathbf{PA}$ *to* $\mathcal{N}(\mathbf{Q})$ *and* $\mathcal{R}(\mathbf{Q})$ *holds, respectively,*

$$\mathbf{PA}\big|_{\mathcal{N}(\mathbf{Q})} = \mathbf{O}\big|_{\mathcal{N}(\mathbf{Q})}\,, \qquad \mathbf{PA}\big|_{\mathcal{R}(\mathbf{Q})} = \mathbf{A}\big|_{\mathcal{R}(\mathbf{Q})}\,, \tag{A.3}$$

where $\mathcal{N}(\mathbf{Q}) = \mathcal{U}$*,* $\mathcal{R}(\mathbf{Q}) = (\mathbf{A}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{U}})^{\perp}$ *and, if* $\mathbf{A}$ *and* $\mathbf{B}$ *commute,* $\mathcal{R}(\mathbf{Q}) = (\mathbf{B}^{\mathsf{H}}\widetilde{\mathcal{Z}})^{\perp}$*.*

Proof. The equation on the left-hand side of (A.3) holds since $\mathcal{N}(\mathbf{PA}) = \mathcal{N}(\mathbf{Q})$, see Statement 20 of Lemma A.3. For proving the one on the right-hand side, let $\mathbf{x} = \mathbf{Qc} \in \mathcal{R}(\mathbf{Q})$ and use the equation $\mathbf{PAQ} = \mathbf{AQ}$ (from Statement 9 of Lemma A.3) to verify that

$$\mathbf{PAx} = \mathbf{PAQc} = \mathbf{AQc} = \mathbf{Ax}\,.$$

[8] Recall: if $\mathbf{P}$ is a projection, so is $\mathbf{I} - \mathbf{P}$, and

$$\mathcal{R}(\mathbf{I} - \mathbf{P}) = \mathcal{N}(\mathbf{P})\,, \qquad \mathcal{N}(\mathbf{I} - \mathbf{P}) = \mathcal{R}(\mathbf{P})\,.$$

□

Note that $\mathcal{N}(\mathbf{Q}) \cap \mathcal{R}(\mathbf{Q}) = \{\mathbf{0}\}$ because $\mathbf{Q}$ is a projection.

In summary, for the two cases $\mathbf{B} = \mathbf{I}$ and $\mathbf{B} = \mathbf{A}$ we are mainly interested in, we get, respectively,

$$\mathbf{E_I} := \widetilde{\mathbf{U}}^\mathsf{H}\mathbf{A}\mathbf{U}\,, \qquad \mathbf{M_I} := \mathbf{U}\mathbf{E_I}^{-1}\widetilde{\mathbf{U}}^\mathsf{H}\,, \qquad \mathbf{P_I} := \mathbf{I} - \mathbf{A}\mathbf{M_I}\,, \qquad \mathbf{Q_I} := \mathbf{I} - \mathbf{M_I}\mathbf{A} \tag{A.4}$$

and

$$\mathbf{E_A} := \widetilde{\mathbf{Z}}^\mathsf{H}\mathbf{Z}\,, \qquad \mathbf{M_A} := \mathbf{U}\mathbf{E_A}^{-1}\widetilde{\mathbf{U}}^\mathsf{H}\,, \qquad \mathbf{P_A} := \mathbf{I} - \mathbf{Z}\mathbf{E_A}^{-1}\widetilde{\mathbf{Z}}^\mathsf{H}\,, \qquad \mathbf{Q_A} := \mathbf{I} - \mathbf{M_A}\mathbf{A}^2\,. \tag{A.5}$$

In BiCG and some other Lanczos-type methods we use additionally a Krylov space generated by $\widehat{\mathbf{A}}^\mathsf{H}$. Therefore we also need to apply $\mathbf{P}^\mathsf{H}$ and $\mathbf{Q}^\mathsf{H}$.

Lemma A.5. *The following properties of* $\mathbf{P}^\mathsf{H} = \mathbf{I} - \mathbf{B}^\mathsf{H}\mathbf{M}^\mathsf{H}\mathbf{A}^\mathsf{H}$ *and* $\mathbf{Q}^\mathsf{H} = \mathbf{I} - \mathbf{A}^\mathsf{H}\mathbf{B}^\mathsf{H}\mathbf{M}^\mathsf{H}$ *hold:*

21) $\mathcal{N}(\mathbf{P}^\mathsf{H}) = \mathbf{B}^\mathsf{H}\widetilde{\mathcal{U}} = \mathcal{R}(\mathbf{B}^\mathsf{H}\mathbf{M}^\mathsf{H})$ *;*

22) $\mathcal{N}(\mathbf{Q}^\mathsf{H}) = \mathbf{A}^\mathsf{H}\mathbf{B}^\mathsf{H}\widetilde{\mathcal{U}} = \mathcal{R}(\mathbf{A}^\mathsf{H}\mathbf{B}^\mathsf{H}\mathbf{M}^\mathsf{H})$ *;*

23) $\mathbf{P}^\mathsf{H}$ *is the projection onto* $\mathcal{R}(\mathbf{P}^\mathsf{H}) = \mathcal{N}(\mathbf{M}^\mathsf{H}\mathbf{A}^\mathsf{H}) = [\mathcal{R}(\mathbf{A}\mathbf{M})]^\perp = \mathcal{Z}^\perp$ *along* $\mathcal{N}(\mathbf{P}^\mathsf{H}) = \mathbf{B}^\mathsf{H}\widetilde{\mathcal{U}}$ *;*

24) $\mathbf{Q}^\mathsf{H}$ *is the projection onto* $\mathcal{R}(\mathbf{Q}^\mathsf{H}) = \mathcal{N}(\mathbf{M}^\mathsf{H}) = \mathcal{U}^\perp$ *along* $\mathcal{N}(\mathbf{Q}^\mathsf{H}) = \mathbf{A}^\mathsf{H}\mathbf{B}^\mathsf{H}\widetilde{\mathcal{U}}$ *;*

25) $\mathcal{R}(\mathbf{Q}^\mathsf{H}\mathbf{A}^\mathsf{H}) = \mathcal{R}(\mathbf{Q}^\mathsf{H}) = \mathcal{N}(\mathbf{M}^\mathsf{H}) = \mathcal{U}^\perp$ *;*

26) $\mathcal{N}(\mathbf{Q}^\mathsf{H}\mathbf{A}^\mathsf{H}) = [\mathcal{R}(\mathbf{A}\mathbf{Q})]^\perp = [\mathcal{R}(\mathbf{P}\mathbf{A})]^\perp = \mathbf{B}^\mathsf{H}\widetilde{\mathcal{U}}$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statements 22 and 24 can be reformulated as*

22') $\mathcal{N}(\mathbf{Q}^\mathsf{H}) = \mathbf{B}^\mathsf{H}\widetilde{\mathcal{Z}} = \mathcal{R}(\mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H}\mathbf{M}^\mathsf{H})$ *;*

24') $\mathbf{Q}^\mathsf{H}$ *is the projection onto* $\mathcal{R}(\mathbf{Q}^\mathsf{H}) = \mathcal{N}(\mathbf{M}^\mathsf{H}) = \mathcal{U}^\perp$ *along* $\mathcal{N}(\mathbf{Q}^\mathsf{H}) = \mathbf{B}^\mathsf{H}\widetilde{\mathcal{Z}}$ *.*

Proof. The verification is again straightforward. □

For the biconjugate residual (BiCR) method applied to the dual problem $\mathbf{A}^\mathsf{H}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}}$ we consider two projections $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ as well as a corresponding projected operator $\widetilde{\mathbf{A}}$ that are in another sense dual to $\mathbf{P}$, $\mathbf{Q}$, and $\widehat{\mathbf{A}}$:

$$\widetilde{\mathbf{P}} := \mathbf{I} - \mathbf{A}^\mathsf{H}\widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}\,, \qquad \widetilde{\mathbf{Q}} := \mathbf{I} - \widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H}\,, \tag{A.6}$$

where

$$\widetilde{\mathbf{E}} := \mathbf{U}^\mathsf{H}\mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H}\widetilde{\mathbf{U}}\,, \qquad \widetilde{\mathbf{M}} := \widetilde{\mathbf{U}}\widetilde{\mathbf{E}}^{-1}\mathbf{U}^\mathsf{H}\,, \tag{A.7}$$

and where we have to assume that $\widetilde{\mathbf{E}}$ is nonsingular.

Note that $\widetilde{\mathbf{E}} = \mathbf{E}^\mathsf{H}$, $\widetilde{\mathbf{M}} = \mathbf{M}^\mathsf{H}$ if $\mathbf{A}$ and $\mathbf{B}$ commute, and that $\widetilde{\mathbf{P}} = \mathbf{Q}^\mathsf{H}$ and $\widetilde{\mathbf{Q}} = \mathbf{P}^\mathsf{H}$ if $\mathbf{B} = \mathbf{I}$, which is the case for BiCG. But the statement on the projectors is not true if, *e.g.*, $\mathbf{B} = \mathbf{A}$. We can summarize the properties of $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ in an analog of Lemma A.3, but before we need to write down the analog of Lemma A.2 for $\widetilde{\mathbf{M}}$. The proofs are again left to the reader.

Lemma A.6. *The following properties of* $\widetilde{\mathbf{M}}$ *hold:*

1) $\widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H}\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}$ *;*

2) $\widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}\mathbf{A}^\mathsf{H}\widetilde{\mathbf{M}} = \widetilde{\mathbf{M}}$ *;*

3) $\mathcal{R}(\widetilde{\mathbf{M}}) = \widetilde{\mathcal{U}}$ *;* $\mathcal{N}(\widetilde{\mathbf{M}}) = \mathcal{U}^\perp$ *;*

4) $\mathcal{R}(\widetilde{\mathbf{M}}^\mathsf{H}) = \mathcal{U}$ *;* $\mathcal{N}(\widetilde{\mathbf{M}}^\mathsf{H}) = \widetilde{\mathcal{U}}^\perp$ *;*

5) $\mathcal{N}(\mathbf{A}^\mathsf{H}\widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}) = \mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^\mathsf{H}) = [\mathcal{R}(\mathbf{B}\widetilde{\mathbf{M}}^\mathsf{H})]^\perp = (\mathbf{B}\mathcal{U})^\perp$ *;*

6) $\mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = [\mathcal{R}(\mathbf{A}\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}})]^{\perp} = (\mathbf{A}\mathbf{B}\mathcal{U})^{\perp}$ *;*

7) $\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{M}}) = \widetilde{\mathcal{Z}}$ *along* $\mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}) = (\mathbf{B}\mathcal{U})^{\perp}$ *;*

8) $\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{M}}) = \widetilde{\mathcal{U}}$ *along* $\mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = (\mathbf{A}\mathbf{B}\mathcal{U})^{\perp}$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statements 6 and 8 can be reformulated as*

6') $\mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = (\mathbf{B}\mathcal{Z})^{\perp}$ *;*

8') $\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{M}}) = \widetilde{\mathcal{U}}$ *along* $\mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = (\mathbf{B}\mathcal{Z})^{\perp}$ *.*

Using these statements we readily obtain the following analog of Lemma A.3.

LEMMA A.7. *The following properties of* $\widetilde{\mathbf{P}}$ *and* $\widetilde{\mathbf{Q}}$ *hold:*

9) $\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}} = \mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}} = \widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}}$ *;*

10) $\mathsf{rank}\,\widetilde{\mathbf{P}} = \mathsf{rank}\,\widetilde{\mathbf{Q}} = N - k$ *;*

11) $\widetilde{\mathbf{Q}}\widetilde{\mathbf{U}} = \mathbf{O}$ *and* $\widetilde{\mathbf{Q}}\widetilde{\mathbf{M}} = \mathbf{O}$ *;*

12) $\widetilde{\mathbf{P}}\widetilde{\mathbf{Z}} = \widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{U}} = \mathbf{O}$ *and* $\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{M}} = \mathbf{O}$ *;*

13) $\mathbf{U}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\widetilde{\mathbf{P}} = \mathbf{O}$ *and* $\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\widetilde{\mathbf{P}} = \mathbf{O}$ *;*

14) $\mathbf{U}^{\mathsf{H}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}} = \mathbf{O}$ *and* $\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}} = \mathbf{O}$ *;*

15) $\mathcal{N}(\widetilde{\mathbf{P}}) = \widetilde{\mathcal{Z}} = \mathcal{R}(\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{M}})$ *;*

16) $\mathcal{N}(\widetilde{\mathbf{Q}}) = \widetilde{\mathcal{U}} = \mathcal{R}(\widetilde{\mathbf{M}})$ *;*

17) $\widetilde{\mathbf{P}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{P}}) = \mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}) = [\mathcal{R}(\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}})]^{\perp} = (\mathbf{B}\mathcal{U})^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{P}}) = \widetilde{\mathcal{Z}}$ *;*

18) $\widetilde{\mathbf{Q}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{Q}}) = \mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = [\mathcal{R}(\mathbf{A}\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}})]^{\perp} = (\mathbf{A}\mathbf{B}\mathcal{U})^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{Q}}) = \widetilde{\mathcal{U}}$ *;*

19) $\mathcal{R}(\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}) = \mathcal{R}(\widetilde{\mathbf{P}}) = \mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}) = (\mathbf{B}\mathcal{U})^{\perp}$ *;*

20) $\mathcal{N}(\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{Q}}) = \widetilde{\mathcal{U}}$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statement 18 can be reformulated as*

18') $\widetilde{\mathbf{Q}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{Q}}) = \mathcal{N}(\widetilde{\mathbf{M}}\mathbf{B}^{\mathsf{H}}\mathbf{A}^{\mathsf{H}}) = (\mathbf{B}\mathcal{Z})^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{Q}}) = \widetilde{\mathcal{U}}$ *.*

In particular, for $\widetilde{\mathbf{A}} := \widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}$ the following theorem holds.

THEOREM A.8. *For the restrictions of* $\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}$ *to* $\mathcal{N}(\widetilde{\mathbf{Q}})$ *and* $\mathcal{R}(\widetilde{\mathbf{Q}})$ *holds, respectively,*

$$\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\big|_{\mathcal{N}(\widetilde{\mathbf{Q}})} = \mathbf{O}\big|_{\mathcal{N}(\widetilde{\mathbf{Q}})}\,, \qquad \widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\big|_{\mathcal{R}(\widetilde{\mathbf{Q}})} = \mathbf{A}^{\mathsf{H}}\big|_{\mathcal{R}(\widetilde{\mathbf{Q}})}\,, \tag{A.8}$$

where $\mathcal{N}(\widetilde{\mathbf{Q}}) = \widetilde{\mathcal{U}}$*,* $\mathcal{R}(\widetilde{\mathbf{Q}}) = (\mathbf{A}\mathbf{B}\mathcal{U})^{\perp}$ *and, if* $\mathbf{A}$ *and* $\mathbf{B}$ *commute,* $\mathcal{R}(\widetilde{\mathbf{Q}}) = (\mathbf{B}\mathcal{Z})^{\perp}$*.*

Proof. Analogous to the proof of Theorem A.4. The equation on the left-hand side of (A.8) holds since $\mathcal{N}(\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{Q}})$, see Statement 20 of Lemma A.7. For proving the one on the right-hand side, let $\widetilde{\mathbf{x}} = \widetilde{\mathbf{Q}}\widetilde{\mathbf{c}} \in \mathcal{R}(\widetilde{\mathbf{Q}})$ and use the equation $\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}} = \mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}}$ (from Statement 9 of Lemma A.7) to verify that

$$\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}}\widetilde{\mathbf{c}} = \mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}}\widetilde{\mathbf{c}} = \mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{x}}\,.$$

□

Finally, we use the Krylov space generated by $\widetilde{\mathbf{A}}^{\mathsf{H}}$, for which we need to apply $\widetilde{\mathbf{P}}^{\mathsf{H}}$ and $\widetilde{\mathbf{Q}}^{\mathsf{H}}$.

LEMMA A.9. *The following properties of* $\widetilde{\mathbf{P}}^{\mathsf{H}} = \mathbf{I} - \mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}}\mathbf{A}$ *and* $\widetilde{\mathbf{Q}}^{\mathsf{H}} = \mathbf{I} - \mathbf{A}\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}}$ *hold:*

21) $\mathcal{N}(\widetilde{\mathbf{P}}^{\mathsf{H}}) = \mathbf{B}\mathcal{U} = \mathcal{R}(\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}})$ *;*

22) $\mathcal{N}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathbf{A}\mathbf{B}\mathcal{U} = \mathcal{R}(\mathbf{A}\mathbf{B}\widetilde{\mathbf{M}}^{\mathsf{H}})$ *;*

23) $\widetilde{\mathbf{P}}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{P}}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{M}}^{\mathsf{H}}\mathbf{A}) = [\mathcal{R}(\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{M}})]^{\perp} = \widetilde{\mathcal{Z}}^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{P}}^{\mathsf{H}}) = \mathbf{B}\mathcal{U}$ *;*

24) $\widetilde{\mathbf{Q}}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{M}}^{\mathsf{H}}) = \widetilde{\mathcal{U}}^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathbf{AB}\mathcal{U}$ *;*

25) $\mathcal{R}(\widetilde{\mathbf{Q}}^{\mathsf{H}}\mathbf{A}) = \mathcal{R}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{M}}^{\mathsf{H}}) = \widetilde{\mathcal{U}}^{\perp}$ *;*

26) $\mathcal{N}(\widetilde{\mathbf{Q}}^{\mathsf{H}}\mathbf{A}) = [\mathcal{R}(\mathbf{A}^{\mathsf{H}}\widetilde{\mathbf{Q}})]^{\perp} = [\mathcal{R}(\widetilde{\mathbf{P}}\mathbf{A}^{\mathsf{H}})]^{\perp} = \mathbf{B}\mathcal{U}$ *.*

If $\mathbf{A}$ *and* $\mathbf{B}$ *commute, then Statements 22 and 24 can be reformulated as*

22') $\mathcal{N}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathbf{B}\mathcal{Z} = \mathcal{R}(\mathbf{BA}\widetilde{\mathbf{M}}^{\mathsf{H}})$ *;*

24') $\widetilde{\mathbf{Q}}^{\mathsf{H}}$ *is the projection onto* $\mathcal{R}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathcal{N}(\widetilde{\mathbf{M}}^{\mathsf{H}}) = \widetilde{\mathcal{U}}^{\perp}$ *along* $\mathcal{N}(\widetilde{\mathbf{Q}}^{\mathsf{H}}) = \mathbf{B}\mathcal{Z}$ *.*

Proof. Again, the verification is straightforward. □

Recall that Table 2.1 of the paper gives an overview of the projections and projected operators that play a key role in this paper. The properties listed there can all be extracted from the results given here.