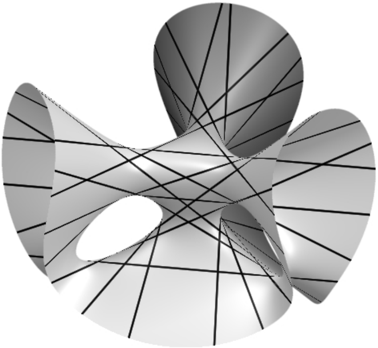
Counting curves in space

maps or equations?



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Enumerative geometry

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3. How many conics does a generic quintic 3-fold contain?
   Ans: 609250 (S. Katz – 1986)

Enumerative geometry

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Ans: $N_1 = N_2 = 1, N_3 = 12, N_4 = 620, \ldots$

$$N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

(Kontsevich – 1994)

One problem, two solutions

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Two ways to think of curves:

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x : y] &\mapsto [x : y : 0] \end{aligned}$$

$$\begin{aligned} \mathcal{I} &\subseteq \mathcal{O}_{\mathbb{P}^2} \\ \mathcal{I} &= (z = 0) \end{aligned}$$

Stable maps and Gromov-Witten theory

The first approach (curves=maps) lead to the development of Gromov-Witten theory in the 90s. Gromov-Witten theory uses the space of stable maps

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$$\overline{M}_{g,m}(X,\beta) = \{(C, p_1, \ldots, p_m, f)\}$$

parametrizing maps $f : C \to X$ from a nodal curve of genus g to X such that $f_*[C] = \beta \in H_2(X)$ and distinct marked points $p_1, \ldots, p_m \in C$.

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But sometimes the spaces $\overline{M}_{g,m}(X,\beta)$ are very singular, sometimes they have strata with higher dimension than expected, etc.

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This homology class lives in degree equal to the expected dimension

$$\mathrm{virdim} = (\dim(X) - 3)(1 - g) + \int_{\beta} c_1(X) + m.$$

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$$Z^X_{\mathrm{GW}} = \exp\left(\sum_{g,\beta} \mathrm{GW}^X_{g,\beta} u^{2g-2} z^\beta\right).$$

Ideal sheaves and Donaldson-Thomas invariants

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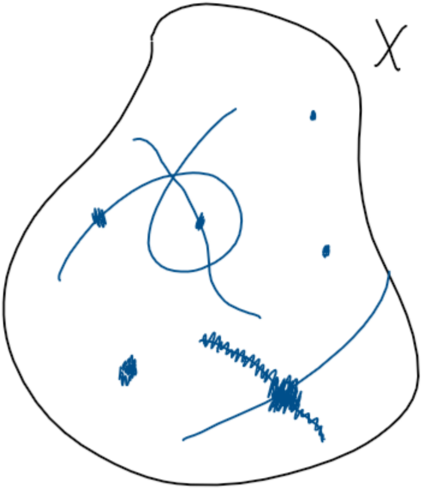
When X is a 3-fold it admits a virtual fundamental class $[I_n(X,\beta)]^{\mathrm{vir}}$. If moreover X is Calabi-Yau, the expected dimension is zero and we define DT invariants

$$\mathrm{DT}^X_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

A picture



A Miró picture

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Theorem (Behrend-Fantechi, Li)

For $\beta = 0$

$$\sum_{n\geq 0} \mathrm{DT}_{n,0}^X q^n = \prod_{k\geq 1} (1-(-q)^k)^{-k\cdot e(X)}$$

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Definition

A stable pair is a sheaf F of *pure* dimension 1 together with a map $\phi : \mathcal{O}_X \to F$ such that *coker* ϕ has dimension 0. Let $P_n(X, \beta)$ be the moduli of stable pairs with $n = \chi(F)$, $\beta = [\mathrm{supp}(F)]$.

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Think of stable pairs as a curve together with points *on the curve*. If $C \subseteq X$ is smooth then stable pairs supported on C are

$$\mathcal{O}_X \to \mathcal{O}_C(D)$$

with $D \subseteq C$ effective divisor.

Pandharipande-Thomas invariants

As before we define the PT invariants and the PT partition function:

$$\mathrm{PT}^X_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

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Stable pairs have a strinking rationality property:

Theorem (Bridgeland 2016)

For every $\beta \in H_2(X)$ the generating function

$$\mathrm{PT}^X_\beta = \sum_{n \in \mathbb{Z}} \mathrm{PT}^X_{n,\beta} q^n$$

is the Laurent expansion of a rational function satisfying the symmetry

$$\mathrm{PT}^X_\beta(q) = \mathrm{PT}^X_\beta(q^{-1}).$$

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The equivalence with Gromov-Witten is more complicated and still conjectural:

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande 2006)

After the change of variables $-q = e^{iu}$ *we have*

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This opens a very interesting direction: we can use the equations side to study/compute the maps side!

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- Motivic description/wall-crossing techniques.

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5. Formulation of the Virasoro constraints on the PT world (M-Oblomkov-Okounkov-Pandharipande, 2020).

3-folds containing $\mathbb{P}^1 \times \mathbb{P}^1$

Let X be a Calabi-Yau 3-fold containing a smooth divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $B \in H_2(X)$ be the curve class of $\mathbb{P}^1 \times \mathrm{pt}$ (and assume the ray generated by B is extremal in the curve cone of X).

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Theorem (Buelles-M, 2021)

Let $\beta \in H_2(X)$, $g \geq 0$. Assume GW/PT correspondence holds. Then

$$\sum_{j \in \mathbb{Z}} \mathrm{GW}^X_{g,\beta+jB} Q^j$$

is the expansion of a rational function $f(Q)$ satisfying

$$f(Q^{-1}) = Q^{-E\cdot\beta} f(Q).$$

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Suggested by physics as consequence of heterotic string+mirror symmetry.

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is the expansion of rational function in $\mathbb{Q}(q, Q)$ satisfying similar symmetry.

The symmetry is explained by a certain automorphism in the derived category

$$\rho = \mathrm{ST}_{\mathcal{O}_E(-C)} \circ \mathrm{ST}_{\mathcal{O}_E(-C+B)} \circ \mathbb{D} \in \mathsf{Aut}(D^b(X)).$$

Thank you!