Final project for the course Lie Groups and Lie Algebras

ON LIE'S THIRD THEOREM

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1. Introduction

Lie's third theorem is one of the fundamental theorems on Lie theory that shows the deep connection between Lie groups and Lie algebras – it says that every finite dimensional Lie algebra $\mathfrak{g}$ is the Lie algebra of some Lie group G; in this case we say that G integrates $\mathfrak{g}$. Note that by taking the universal cover of G Lie's third theorem is equivalent to saying that there is a simply connected Lie group integrating $\mathfrak{g}$. In connection with Lie's second theorem it shows that the categories of finite dimensional Lie algebras and simply connected Lie groups are equivalent. The original Lie's third theorem, proven by Sophus Lie, was only local (in the sense it said that a Lie algebra integrated to a local Lie group); only later Cartan proved the modern version.

There are several proofs of Lie's third theorem. The more common one uses Ado's theorem, that states that every finite dimensional Lie algebra is a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V, and hence it integrates to some subgroup of $\mathrm{GL}(V)$. However, Ado's theorem is hard and requires a lot of the structure theory of Lie algebras.

In this paper we will show a different, more geometric, and not so well known proof of Lie's third theorem. The proof will be based on the one in [3]. This proof is much more constructive than others, which frequently use some sort of induction and classification results on Lie algebras – it describes the Lie group integrating $\mathfrak{g}$ as a quotient of the path space of $\mathfrak{g}$ by a certain subgroup. Because of this, the proof we will present gives a better understanding of the integrability problem. Indeed, a similar approach can be used to understand the integrability problem for more general objects, for instance for Banach Lie algebras (see [4]) and Lie algebroids (see [1] and [2]), in which integrability is not always possible but the obstructions to it can be understood.

The paper is organized as follows: In section 2 we explain the construction of the Lie group integrating $\mathfrak{g}$ as a quotient of the path space of $\mathfrak{g}$; at the end we will also take the opportunity to give a different proof of Lie's second theorem. Since the path space of $\mathfrak{g}$ is a Banach Lie group, in section 3 we discuss a bit about the general theory of such infinite dimensional Lie groups and state the results which will be useful. In section 4 we prove that the construction in 2 gives actually a Lie group with Lie algebra $\mathfrak{g}$, proving Lie's third theorem. In section 5 we show an example of a Banach Lie algebra which doesn't integrate to a Lie group and discuss briefly the integration problem for Banach Lie algebras. At last, in 6 we mention a few other possible proofs and related results.

2. Construction of Lie group integrating $\mathfrak{g}$

Given a connected Lie group G, we can describe its universal cover as paths in G starting at 1 modulo homotopies that fix the end points. More precisely we let

$$P(G) = \{\gamma \in C^1([0,1], G) : \gamma(0) = 1 \in G\}$$

and define an equivalence relation in $P(G)$ by $\gamma_0 \sim \gamma_1$ if and only if there is a C^1 homotopy γ_ϵ such that $\gamma_\epsilon(1)$ and $\gamma_\epsilon(0)$ are fixed (in particular this implies that if $\gamma_0 \sim \gamma_1$ then $\gamma_0(1) = \gamma_1(1)$). Note that $P(G)$ can be given the C^1 topology and can be endowed with a product defined pointwise, that is, the product of $\gamma, \gamma' \in P(G)$ is defined by $(\gamma\gamma')(t) = \gamma(t)\gamma'(t)$ – this makes $P(G)$ a topological group. Note also that $\sim$ is well behaved with respect to the product in $P(G)$, since if $\gamma_\epsilon, \gamma'_\epsilon$ are homotopies between $\gamma_0 \sim \gamma_1$ and $\gamma'_0 \sim \gamma'_1$, respectively, then $\gamma_\epsilon\gamma'_\epsilon$ is a homotopy between $\gamma_0\gamma'_0$ and $\gamma_1\gamma'_1$. Then the universal cover $\widetilde{G}$ of G is (as a topological group)

$$\widetilde{G} = P(G)/\sim = P(G)/P(G)_0$$

where $P(G)_0 = \{\gamma \in P(G) : \gamma \sim 1\}$ where 1 is the constant path. Note that this is precisely the usual construction of the universal cover, except that we're requiring paths and homotopies to be C^1 instead of just continuous, which is fine since we can approximate continuous paths and homotopies by smooth ones.

The main idea of the construction is that this description of the universal cover of G can actually (as is expected, knowing Lie's third theorem) be written completely in terms of the Lie algebra $\mathfrak{g}$ of G. To do this, let

$$P(\mathfrak{g}) = C^0([0,1], \mathfrak{g})$$

and define $D : P(G) \to P(\mathfrak{g})$ by

$$(D\gamma)(t) = (dR_{\gamma(t)^{-1}})\dot{\gamma}(t) = \frac{d}{ds}\left(\gamma(s)\gamma(t)^{-1}\right)_{s=t}.$$

The map D, which should be seen as differentiation of paths, relates paths in G with paths in $\mathfrak{g}$. Indeed, D is a homeomorphism.

Lemma 1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the map D is a homeomorphism $P(G) \to P(\mathfrak{g})$.*

Proof. By the definitions of the C^1 and C^0 topology it's clear that D is continuous. Given $\delta \in P(\mathfrak{g})$ denote by $F_\delta : G \times [0,1] \to TG$ the map

$$F_\delta(g,t) = (dR_g)\delta(t).$$

Since $F_\delta(g,t) \in T_gG$ we can regard F_δ as a time dependent vector field on G. Note that $\delta = D\gamma$ if and only if

$$\dot{\gamma}(t) = F_\delta(\gamma(t), t)$$

and $\gamma(0) = 1$. This already makes clear that D is injective, by uniqueness of solutions to ODEs. To show that D is surjective we need to see that the maximal interval of definition of the solution of the above ODE is $[0,1]$. For fixed δ, let $\Phi^{s,t}(x)$ be the flow of F_δ which is x at time s; then $\gamma(t) = \Phi^{0,t}(1_G)$ is a solution of $D\gamma = \delta$.

Let I denote the set of pairs (s,t) in $[0,1]$ for which $\Phi^{s,t}(1)$ is defined; certainly $(s,s) \in I$ for any $s \in [0,1]$ and by the existence theorem for ODEs I_s is open for each s. An easy computation shows that $\Phi^{s,t}(x) = \Phi^{s,t}(1)x$, hence $\Phi^{s,t}(x)$ is defined for every $(s,t) \in I$. Moreover since $\Phi^{s,u} \circ \Phi^{u,t} = \Phi^{s,t}$ we get that if $(s,u),(u,t) \in I$ then $(s,t) \in I$. With this, it's easy to see that $I = [0,1] \times [0,1]$: by compacity of $[0,1]$ there are $0 = s_1 < s_1 < \ldots < s_k = 1$ such that I_{s_j} cover $[0,1]$ and $s_{j+1} \in I_{s_j}$, so given any $(s,t) \in [0,1] \times [0,1]$ wlog such that $s < t$ we can find i,j such that $(s,s_i),(s_i,s_{i+1}),\ldots,(s_j,t)$, and thus $(s,t) \in I$. In particular $(0,t) \in I$ for every $t \in [0,1]$, thus D is surjective.

The fact that D^{-1} is continuous follows from the fact that the map $\delta \mapsto F_\delta$ from $P(\mathfrak{g})$ to the space of time dependent vector fields is continuous and from the continuous dependence of the solution of an ODE on the vector field. □

To understand the construction in terms of the Lie algebra $\mathfrak{g}$ we must see to what do the product and the equivalence relation $\sim$ in $P(G)$ correspond after applying the map D. To describe this, the following definition is useful.

Definition 1. *Given $\delta \in P(\mathfrak{g})$, let $A = A_\delta : [0,1] \to \mathrm{GL}(\mathfrak{g})$ be the solution of the differential equation*

$$\begin{cases} \frac{dA}{dt}(t) = \operatorname{ad}\delta(t) \circ A(t) \\ A(0) = I. \end{cases}$$

Alternatively, $A_\delta = D^{-1}(\operatorname{ad}\delta)$ where $D : P(\operatorname{Ad}\mathfrak{g}) \to P(\operatorname{ad}\mathfrak{g})$.

Lemma 2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then*

(1) For any $\gamma, \gamma' \in P(G)$ we have

$$D(\gamma\gamma')(t) = D\gamma(t) + \operatorname{Ad}(\gamma(t))(D\gamma')(t)$$

and moreover $\operatorname{Ad}(\gamma(t)) = A_{D\gamma}(t)$.

(2) A smooth homotopy $\epsilon \mapsto \gamma_\epsilon \in P(G)$ has fixed end point (that is, $\gamma_\epsilon(1)$ is constant) if and only if

$$\int_0^1 A_{\delta_\epsilon}(t)^{-1}\,\frac{\partial\delta_\epsilon(t)}{\partial\epsilon}\,dt = 0. \tag{1}$$

where $\delta_\epsilon = D\gamma_\epsilon$. In particular $P(\mathfrak{g})_0 \equiv D(P(G)_0)$ is the set of $\delta \in P(\mathfrak{g})$ such that there is a map $\epsilon \mapsto \delta_\epsilon$ obeying equation such that $\delta_0 = \delta$ and $\delta_1 = 0$

Proof. (1) We compute the following

$$\begin{aligned}\frac{d}{dt}(\gamma(t)\gamma'(t)) &= (dR_{\gamma'(t)})\dot{\gamma}(t) + (dL_{\gamma(t)})\dot{\gamma}'(t)\\ &= (dR_{\gamma'(t)} \circ dR_{\gamma(t)})(D\gamma(t)) + (dL_{\gamma(t)} \circ dR_{\gamma'(t)})(D\gamma')(t)\\ &= (dR_{\gamma(t)\gamma'(t)})\left(D\gamma(t) + d(R_{\gamma(t)^{-1}} \circ L_{\gamma(t)})D\gamma'(t)\right)\\ &= (dR_{\gamma(t)\gamma'(t)})\left(D\gamma(t) + \mathrm{Ad}(\gamma(t))D\gamma'(t)\right).\end{aligned}$$

Hence the formula for $D(\gamma\gamma')$ follows. To show that $\mathrm{Ad}(\gamma(t)) = A_{D\gamma}(t)$ we can compute the derivative

$$\begin{aligned}\frac{d}{ds}\mathrm{Ad}(\gamma(s))_{s=t} &= \left(\frac{d}{ds}\mathrm{Ad}(\gamma(s)\gamma(t)^{-1})\right) \circ \mathrm{Ad}(\gamma(t))\\ &= \mathrm{ad}\left(\frac{d}{ds}\gamma(s)\gamma(t)^{-1}\right) \circ \mathrm{Ad}(\gamma(t))\\ &= \mathrm{ad}(D\gamma(t)) \circ \mathrm{Ad}(\gamma(t)).\end{aligned}$$

Since moreover $\mathrm{Ad}(\gamma(0)) = \mathrm{Ad}(1) = I$ the result follows.

(2) To prove this we fix ϵ and t and consider the function $F : [0,1] \times [0,1] \to G$ defined by

$$F(s,u) = \gamma_\epsilon(s)^{-1}\gamma_u(s)\gamma_u(t)^{-1}\gamma_\epsilon(t).$$

Note that $F(t,u) = 1 = F(s,\epsilon)$ for any $u, s \in [0,1]$. Hence (t,ϵ) is a critical point of F, and therefore its Hessian is well defined and symmetric. We compute its cross derivatives at (t,ϵ):

$$\begin{aligned}\frac{\partial^2 F}{\partial u \partial s}(t,\epsilon) &= \frac{\partial}{\partial u}\left(\frac{\partial}{\partial s}\left(\gamma_\epsilon(t)^{-1}\gamma_u(s)\gamma_u(t)^{-1}\gamma_\epsilon(t)\right)_{s=t} + \frac{\partial}{\partial s}\left(\gamma_\epsilon(s)^{-1}\gamma_\epsilon(t)\right)_{s=t}\right)_{u=\epsilon}\\ &= \frac{\partial}{\partial u}\left(\frac{\partial}{\partial s}\left(\gamma_\epsilon(t)^{-1}\gamma_u(s)\gamma_u(t)^{-1}\gamma_\epsilon(t)\right)_{s=t}\right)_{u=\epsilon}\\ &= (dL_{\gamma_\epsilon(t)})^{-1}(dR_{\gamma_\epsilon(t)})\frac{\partial}{\partial u}\left(\frac{\partial}{\partial s}\left(\gamma_u(s)\gamma_u(t)^{-1}\right)_{s=t}\right)_{u=\epsilon}\\ &= \mathrm{Ad}(\gamma_\epsilon(t))^{-1}\frac{\partial \delta_\epsilon(t)}{\partial \epsilon}.\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{\partial^2 F}{\partial s \partial u}(t,\epsilon) &= \frac{\partial}{\partial s}\left(\frac{\partial}{\partial u}\left(\gamma_\epsilon(s)^{-1}\gamma_u(s)\right)_{u=\epsilon} + \frac{\partial}{\partial u}\left(\gamma_u(t)^{-1}\gamma_\epsilon(t)\right)_{u=\epsilon}\right)_{s=t}\\ &= \frac{\partial}{\partial s}\left((dL_{\gamma_\epsilon(s)})^{-1}\frac{\partial \gamma_\epsilon(s)}{\partial \epsilon}\right)_{s=t}.\end{aligned}$$

Using $\frac{\partial^2 F}{\partial u \partial s}(t,\epsilon) = \frac{\partial^2 F}{\partial s \partial u}(t,\epsilon)$ and integrating from $t=0$ to 1 gives

$$\int_0^1 \mathrm{Ad}(\gamma_\epsilon(t))^{-1}\frac{\partial \delta_\epsilon(t)}{\partial \epsilon}\,dt = \left(dL_{\gamma_\epsilon(1)}\right)^{-1}\frac{\partial \gamma_\epsilon(1)}{\partial \epsilon}.$$

Since $\mathrm{Ad}(\gamma_\epsilon(t)) = A_{\delta_\epsilon}(t)$ as we showed in (1) the result follows. $\square$

According to this, given an abstact Lie algebra g we can perform this exact construction:

Definition 2. *Given a Lie algebra $\mathfrak{g}$ we can give the Banach space $P(\mathfrak{g})$ a product structure by defining*

$$(\delta\delta')(t) = \delta(t) + A_\delta(t)\delta'(t).$$

We also define $P(\mathfrak{g})_0$ to be the set of $\delta \in P(\mathfrak{g})$ such that there is a differentiable map $\epsilon \mapsto \delta_\epsilon$ obeying equation 2 such that $\delta_0 = \delta$ and $\delta_1 = 0$.

Lemma 3. *The product defined above gives a group structure on $P(\mathfrak{g})$. Moreover, $\delta \mapsto A_\delta$ is a group homomorphism $P(\mathfrak{g}) \to \mathrm{Ad}\,\mathfrak{g}$.*

Proof. We prove first that $A_\delta A_{\delta'} = A_{\delta\delta'}$. Indeed, by the Leibniz rule

$$\begin{aligned}\frac{d}{dt}(A_\delta A_{\delta'})(t) &= A_\delta(t)\mathrm{ad}(\delta'(t))A_{\delta'}(t) + \mathrm{ad}(\delta(t))A_\delta(t)A_{\delta'}(t)\\ &= \mathrm{ad}(\delta(t) + A_\delta(t)\delta'(t))A_\delta(t)A_{\delta'(t)}\\ &= \mathrm{ad}((\delta\delta')(t))A_\delta(t)A_{\delta'(t)}.\end{aligned}$$

Hence $A_\delta A_{\delta'} = A_{\delta\delta'}$. Note that we used that $\mathrm{ad}(AX) = A\mathrm{ad}(X)A^{-1}$ where $A = A_\delta(t)$ is a Lie homomorphism and $\delta'(t) \in \mathfrak{g}$. Associativity is a matter of computations using the above, showing that

$$\begin{aligned}(\delta_1(\delta_2\delta(3))(t) &= \delta_1(t) + A_{\delta_1}(t)\delta_2(t) + A_{\delta_1}(t)A_{\delta_2}(t)\delta_3(t)\\ &= ((\delta_1\delta_2)\delta_3(t).\end{aligned}$$

The identity is the constant path equal to 0, and it's easy to check that

$$\delta^{-1}(t) = -A_\delta(t)^{-1}\delta(t)$$

defines an inverse of δ. $\square$

With all the results proven, the following is now clear:

Proposition 4. *If $\mathfrak{g}$ is the Lie algebra of G, then $P(\mathfrak{g})/P(\mathfrak{g})_0 \cong \widetilde{G}$.*

Proof. The homeomorphism D^{-1} induces an isomorphism

$$P(\mathfrak{g})/P(\mathfrak{g})_0 \cong P(G)/P(G)_0 \cong \widetilde{G}. \quad \square$$

At this point, we take the opportunity to give a proof of Lie's second theorem with the interpretation of the simply connected group G with Lie algebra $\mathfrak{g}$ which we've been describing. Indeed, with this setup the proof is very natural: given a homomorphism of Lie algebras, the pushforward will give a homomorphism on the path spaces, which will induce a homomorphism from G.

Theorem 5 (Lie II). *Let G be a simply-connected Lie group and H a connected Lie group, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism then there is a unique Lie group homomorphism $f : G \to H$ such that $\phi = (df)_e$.*

Proof. We may assume that H is simply-connected as well, otherwise we just have to compose with the projection from the universal cover $\widetilde{H} \to H$. But then we can define $f = \phi_* : P(\mathfrak{g}) \to P(\mathfrak{h})$ by $f(\delta) = \phi \circ \delta$. Since ϕ is a Lie algebra morphism we have that $\operatorname{ad} f(\delta) \circ \phi = \phi \circ \operatorname{ad} \delta$, and therefore $A_{f(\delta)} \circ \phi = \phi \circ A_\delta$. With this we show, from lemma 2, that f is a Lie group homomorphism and that if $\delta_0 \sim \delta_1$ then $f(\delta_0) \sim f(\delta_1)$, and therefore f induces a map $f : G = P(\mathfrak{g})/P(\mathfrak{g})_0 \to P(\mathfrak{h})/P(\mathfrak{h})_0 = H$ which is a homomorphism.

To see that f lifts ϕ, we compute the exponential $\mathfrak{g} \to P(\mathfrak{g})/P(\mathfrak{g})_0$. This can be written as the composition

$$\mathfrak{g} \overset{\exp}{\to} G \to P(G)/P(G)_0 \overset{D}{\to} P(\mathfrak{g})/P(\mathfrak{g})_0$$

where the map $G \to P(G)/P(G)_0$ sends $g \in G$ to the class of any path from 1 to g. Given $X \in \mathfrak{g}$, take the path $\gamma(t) = \exp(tX)$ and an easy computation shows that $(D\gamma)(t) = X$. Thus, the expoential when seen as a map $\mathfrak{g} \to P(\mathfrak{g})/P(\mathfrak{g})_0$ takes $X \in \mathfrak{g}$ to the class of the constant path $t \mapsto X$. Therefore

$$\exp((df)_e X) = f(\exp(X)) = [\text{constant path} = \phi(X)] = \exp(\phi(X)).$$

Since the exponential is a local isomorphism $\phi = (df)_e$. □

3. Banach Lie Groups and Lie Algebras

A Banach manifold is a topological space modelled by a Banach space. In a Banach space we have a notion of (Fréchet) derivative generalizing the derivative in $\mathbb{R}^n$, enabling us to define a (differentiable) Banach manifold by asking that the transition maps are differentiable. This also gives a definition of smooth maps between Banach manifolds.

In a Banach manifold the implicit function theorem, existence-unicity theorems for ordinary differential equations and Frobenius theorem on integrable distributions still hold, so a lot of the theory for finite dimensional manifolds works as well in the Banach case. The book [5] develops the theory of manifolds in the Banach setup and is a reference for the results mentioned above.

A Banach Lie group is a Banach manifolds endowed with a group structure for which the product and the inverse are smooth. A Banach Lie group has an associated Lie algebra, defined precisely as in the finite dimensional case, which is now a Banach space with a Lie bracket. In Banach Lie groups we can define an exponential, by existence of solutions to ODEs; this is not true for more general classes of infinite dimensional Lie groups. Moreover, the exponential is a local diffeomorphism onto an open set; again, this is not true for other classes of Lie groups (even when we can define an exponential), for instance it fails for $\operatorname{Diff}(S^1)$.

A reference for infinite dimensional Lie groups (and in particular for Banach ones) is [8]. We now define subgroups of Banach Lie groups in a similar way to [4], although with different names:

Definition 3. *Let G be a Banach Lie group. A Lie subgroup of G is a Banach Lie subgroup H admitting an injective inclusion map $H \hookrightarrow G$ such that the induced map on the Lie algebras is an embedding.*

If the inclusion $H \hookrightarrow G$ is an embedding we say that H is an embedded Lie subgroup.

Recall that we want to show that we can give a smooth structure to $P(\mathfrak{g})/P(\mathfrak{g})_0$. The following result shows that to prove this it's enough to see that $P(\mathfrak{g})_0$ is an embedded Lie subgroup, which will be the last part of the proof.

Theorem 6. *If G is a Banach Lie group and N is a normal embedded Lie group of G then G/N can be given a Banach Lie group structure compatible with the quotient topology. Moreover $\pi : G \to G/N$ is a submersion and a fiber bundle, and the Lie algebra of G/N is identified with $\mathfrak{g}/\mathfrak{n}$ where $\mathfrak{g}$ and $\mathfrak{n}$ are the Lie algebras of G and N.*

Proof. Everything except the fiber bundle part is proven in [4] (theorem II.2); we will prove the remaining.

Take $x \in P(\mathfrak{g})$. By the normal form of a submersion for Banach manifolds (see [5]) there are open sets U, V such that $x \in U \subseteq G$, $\pi(x) \in V = \pi(U) \subseteq G/N$ and a splitting map $\phi : V \to U$ which is an embedding of V into U, $x \in \operatorname{im} \phi$ and $\pi \circ \phi = \mathrm{id}_V$. But then the map $N \times V \to \pi^{-1}(V)$ defined by $(y, z) \mapsto y\phi(z)$ is a diffeomorphism, hence V is a local trivialization. □

The closed subgroup theorem, which states that a closed Lie subgroup is an embedded Lie subgroup is not true for Banach Lie groups. However, the following result gives an easy way to show that a subgroup is embedded, and will be how we shall prove that $P(\mathfrak{g})_0$ is embedded.

Lemma 7. *If $f : G \to H$ is a smooth homomorphism of Banach Lie groups and $T \subseteq H$ is an embedded Lie subgroup of H, then $f^{-1}(T) \subseteq G$ is an embedded Lie subgroup of G.*

Proof. See [4], lemma II.1. □

It will also be useful the fact that the second Lie theorem also holds for Banach Lie groups. Indeed, the proof we gave for the finite dimensional case works just as well for the Banach case.

Theorem 8 (Lie II for Banach Lie groups)**.** *Let G be a simply-connected Banach Lie group and H a connected Banach Lie group, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism then there is a unique Lie group homomorphism $f : G \to H$ such that $\phi = (df)_e$.*

4. Proof of Lie III

Note that $P(\mathfrak{g})$ is naturally a vector space, and indeed it's a Banach vector when given the norm sup norm $\|\cdot\|_\infty$. The topology induced by this norm is the same as the C^0 topology we defined $P(\mathfrak{g})$ with (which is the compact-open topology of maps $[0,1]\to\mathfrak{g}$). This makes $P(\mathfrak{g})$ a Banach Lie group with the product defined earlier.

Proposition 9. *The Lie algebra $P(\mathfrak{g})^{\mathrm{alg}}$ of $P(\mathfrak{g})$ can be identified with $P(\mathfrak{g})$ with a Lie bracket given by*

$$[X,Y](t)=\frac{d}{dt}\left[\int_0^t X(s)\,ds,\int_0^t Y(s)\,ds\right].$$

Moreover, $P(\mathfrak{g})_0$ is a connected normal Banach Lie subgroup of $P(\mathfrak{g})$ with corresponding Lie algebra

$$P(\mathfrak{g})_0^{\mathrm{alg}}=\left\{X\in P(\mathfrak{g})^{\mathrm{alg}}:\int_0^1 X(s)\,ds=0\right\}.$$

Proof. Since $P(\mathfrak{g})$ is a vector space, we can identify its Lie algebra with $P(\mathfrak{g})$ itself. We denote by $X(t),Y(t)$ elements of $P^{\mathrm{alg}}(\mathfrak{g})$, which are paths $[0,1]\to\mathfrak{g}$. To prove the formula for the Lie bracket $[X,Y]$ we differentiate $\delta\delta'\delta^{-1}$ with δ, δ' in the directions of X, Y, respectively. We have

$$(\delta\delta'\delta^{-1})(t)=\delta(t)+A_\delta(t)-A_\delta(t)A_{\delta'}(t)A_\delta(t)^{-1}\delta(t).$$

Note that

$$\frac{\partial}{\partial\epsilon}\left(\frac{\partial}{\partial t}A_{\epsilon Y}(t)\right)_{\epsilon=0}=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(A_{\epsilon Y}(t)\circ\mathrm{ad}(\epsilon Y(t))\right)=\mathrm{ad}(Y(t)).$$

Integrating this in t gives

$$\frac{\partial}{\partial\epsilon}A_{\epsilon Y}(t)_{\epsilon=0}=\int_0^t\mathrm{ad}\,Y(s)\,ds.$$

Differentiating $\delta\delta'\delta^{-1}$ with respect to δ' in the direction of Y and using the above gives

$$A_\delta(t)Y(t)-A_\delta(t)\left(\int_0^t\mathrm{ad}\,Y(s)\,ds\right)A_\delta(t)^{-1}\delta(t).$$

Now differentiating with respect to δ in the direction of X gives

$$\begin{aligned}[X,Y](t) &= \left(\int_0^t \operatorname{ad} X(s)\,ds\right) Y(t) - \left(\int_0^t \operatorname{ad} Y(s)\,ds\right) X(t) \\ &= \left[\int_0^t X(s)\,ds, Y(t)\right] + \left[X(t), \int_0^t Y(s)\,ds\right] \\ &= \frac{d}{dt}\left[\int_0^t X(s)\,ds, \int_0^t Y(s)\,ds\right].\end{aligned}$$

Let $\operatorname{av} : P(\mathfrak{g})^{\mathrm{alg}} \to \mathfrak{g}$ be defined by

$$\operatorname{av}(X) = \int_0^1 X(t)\,dt.$$

It's clear that av is a Lie algebra homomorphism, hence

$$A = \ker \operatorname{av} = \left\{X \in P(\mathfrak{g})^{\mathrm{alg}} : \int_0^1 X(s)\,ds = 0\right\}$$

is a Lie ideal (and in particular a sub-algebra) of $P(\mathfrak{g})$. Consider the left invariant distribution on $P(\mathfrak{g})$ given by $\mathcal{D}_\delta = (dL_\delta)A$. This distribution is involutive (because A is a Lie sub-algebra) and the maximal integral submanifold through 0 is a Lie subgroup of $P(\mathfrak{g})$, which we want to show that is $P(\mathfrak{g})_0$.

Note that $(dL_\delta)X(t) = A_\delta(t)X(t)$. Hence, the definition of $P(\mathfrak{g})_0$ says that $\delta \in P(\mathfrak{g})_0$ if and only if there is a path δ_ϵ in $P(\mathfrak{g})$ from δ to 0 such that $(dL_{\delta_\epsilon})^{-1}\frac{d\delta_\epsilon}{d\epsilon} \in A$, that is, such that $\frac{d\delta_\epsilon}{d\epsilon} \in \mathcal{D}$ for every ϵ (i.e. $\epsilon \mapsto \delta_\epsilon$ is tangent to $\mathcal{D}$). This shows that $P(\mathfrak{g})_0$ is the maximal integral submanifold tangent to $\mathcal{D}$, hence it's the connected Lie subgroup of $P(\mathfrak{g})$ with lie algebra $P(\mathfrak{g})_0^{\mathrm{alg}} = A$. Moreover $P(\mathfrak{g})_0$ is normal since its Lie algebra A is a Lie ideal. □

By theorem 6 we now want to show that $P(\mathfrak{g})_0$ is actually an embedded Lie subgroup of $P(\mathfrak{g})$. From this it will follow that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is a Banach Lie group with Lie algebra $P(\mathfrak{g})^{\mathrm{alg}}/P(\mathfrak{g})_0^{\mathrm{alg}}$. Note that av is surjective (since it sends the constant path $X(t) = X \in \mathfrak{g}$ to X), hence av induces an isomorphism of Lie algebras $P(\mathfrak{g})^{\mathrm{alg}}/P(\mathfrak{g})_0^{\mathrm{alg}} \cong \mathfrak{g}$, showing that $P(\mathfrak{g})/P(\mathfrak{g})_0$ is a Lie group integrating $\mathfrak{g}$, and by proposition 4 it follows that it's the unique simply connected group integrating $\mathfrak{g}$.

Our strategy will be to identify $P(\mathfrak{g})_0$ as the kernel of a certain homomorphism ϕ, and by lemma 7 we'll have that $P(\mathfrak{g})_0$ is an embedded Lie subgroup. Since the Lie algebra $P(\mathfrak{g})_0^{\mathrm{alg}}$ is the kernel of av, if we could integrate av we would be able to find such a homomorphism. Although we can't apply this reasoning (for obvious reasons), we can consider the short exact sequence

$$0 \to \mathfrak{z} \to \mathfrak{g} \to \operatorname{ad}\mathfrak{g} \to 0$$

and use this idea with $\operatorname{ad}\mathfrak{g}$, which we know to be integrable to $\operatorname{Ad}\mathfrak{g}$. With the construction we used in our proof of Lie II, we expect that $\operatorname{ad}:\mathfrak{g}\to\operatorname{ad}\mathfrak{g}$ integrates to a homomorphism $P(\mathfrak{g})/P(\mathfrak{g})_0\to\operatorname{Ad}\mathfrak{g}$ given by $[\delta]\mapsto A_\delta(1)$.

Considering all the motivation above, we define

$$P(\mathfrak{g})_1=\{\delta\in P(\mathfrak{g}):A_\delta(1)=I\}=\ker\pi$$

where $\pi:P(\mathfrak{g})\to\operatorname{Ad}(\mathfrak{g})$ is the map $\delta\mapsto A_\delta(1)$; note that π is a Lie group homomorphism (recall that in lemma 3 we showed that $\delta\mapsto A_\delta$ was a homomorphism). Then $P(\mathfrak{g})_1$ is an embedded Lie subgroup by lemma 7. We can compute its Lie algebra:

Proposition 10. *The Lie algebra of* $P(\mathfrak{g})_1$ *is*

$$P(\mathfrak{g})_1^{\mathrm{alg}}=\{X\in P(\mathfrak{g})^{\mathrm{alg}}:\mathrm{av}X\in\mathfrak{z}\}.$$

Proof. The Lie algebra of $P(\mathfrak{g})_1$ consists of the vectors X such that $A_{\epsilon X}(1)=I$ for every ϵ. As we saw in the proof of proposition 9 we have

$$\frac{d}{d\epsilon}A_{\epsilon X}(1)_{\epsilon=0}=\int_0^1\operatorname{ad}X(s)\,ds=\operatorname{ad}(\operatorname{av}X)$$

and this is 0 if and only if $\operatorname{av}(X)\in\mathfrak{z}$. □

Note that in particular we get that $P(\mathfrak{g})_0^{\mathrm{alg}}\subseteq P(\mathfrak{g})_1^{\mathrm{alg}}$, and since $P(\mathfrak{g})_0$ is connected it follows that $P(\mathfrak{g})_0\subseteq P(\mathfrak{g})_1^\circ\subseteq P(\mathfrak{g})_1$ (where we denote by H° the connected component of the identity of a Lie group H).

The map $\pi:P(\mathfrak{g})\to\operatorname{Ad}\mathfrak{g}$ given by $\delta\mapsto A_\delta(1)$ is surjective. Indeed, given $A\in\operatorname{Ad}\mathfrak{g}$ take a path $a:[0,1]\to\operatorname{Ad}\mathfrak{g}$ from I to A. Then $D(a)$ is a path in $\operatorname{ad}\mathfrak{g}$, so we can find $\delta\in P(\mathfrak{g})$ such that $D(a)(t)=\operatorname{ad}(\delta(t))$ and then clearly $A=A_\delta(1)=\pi(\delta)$. Hence we have a fibration

$$\begin{array}{ccc}P(\mathfrak{g})_1 & \longrightarrow & P(\mathfrak{g})\\ & & \big\downarrow{\scriptstyle\pi}\\ & & \operatorname{Ad}\mathfrak{g}.\end{array}$$

This fibration gives an isomorphism $\operatorname{Ad}\mathfrak{g}\cong P(\mathfrak{g})/P(\mathfrak{g})_1$, so by theorem 6 π is a fiber bundle. Since $P(\mathfrak{g})$ is contractible (it's a vector space, so it admits a linear deformation retract to 0) the long exact sequence on homotopy groups gives an isomorphism

$$\pi_1(P(\mathfrak{g})_1,0)\cong\pi_2(\operatorname{Ad}\mathfrak{g},I).$$

But π_2 of a (finite dimensional) Lie group is trivial (this follows from theorem 21.7 in [7]), so $\pi_1(P(\mathfrak{g})_1,0)=0$ and thus $P(\mathfrak{g})_1^\circ$ is simply connected. Now note that av restricts to a map $P(\mathfrak{g})_1^{\mathrm{alg}}\to\mathfrak{z}$, hence by the second Lie theorem (for Banach Lie groups) we can integrate av to a Lie group homomorphism $\phi:P(\mathfrak{g})_1^\circ\to Z$ where $Z=(\mathbb{R}^n,+)$, with $n=\dim\mathfrak{z}$, is the Lie group integrating $\mathfrak{z}$. But then

$$(\ker\phi)^{\mathrm{alg}}=\ker(d\phi)_e=\ker\,\mathrm{av}=P(\mathfrak{g})_0^{\mathrm{alg}}$$

and therefore $P(\mathfrak{g})_0 = (\ker\phi)^\circ$. By lemma 7 we know that $\ker\phi$ is an embedded Lie subgroup, and then so is $P(\mathfrak{g})_0$, finishing the proof of Lie III.

5. Counter example for infinite dimensional Lie groups

The first example of a non-integrable Banach Lie algebra was given by van Est in 1964. We will show an example, given by Serre in [10], of a Banach Lie algebra which doesn't integrate to a Lie group. Let $G = \mathrm{GL}(H) \times \mathrm{GL}(H)$ where H is a complex Hilbert space. The center of G is $Z(G) = \mathbb{C}^* \times \mathbb{C}^*$; let

$$N = \{(e^{it}, e^{\alpha it}) : t \in \mathbb{R}\} \subseteq Z(G)$$

and let $\mathfrak{n}$ be its corresponding Lie algebra. Since N is normal, $\mathfrak{n}$ is an ideal so we can define the Lie algebra $\mathfrak{g}/\mathfrak{n}$, which we claim to be non-integrable.

Suppose that $\mathfrak{g}/\mathfrak{n}$ integrates to a simply connected Lie group K. By Lie's second theorem the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ integrates to some map $\phi : \mathrm{GL}(H)\times\mathrm{GL}(H) \to K$, and we can easily check that this implies that N is $(\ker\phi)^\circ$, hence N would be closed, which is not true when $\alpha \notin \mathbb{Q}$.

We also remark that this example, and in general the failure of Lie's third theorem for infinite dimensional Lie algebras (or for Lie algebroids, as studied in [1] and [2]), is related to the fact that $\pi_2(G)$ (or $H^2(G;\mathbb{R})$) may not be trivial in such cases. An example of this fact is given as above when $\alpha \in \mathbb{Q}$; in this case $K = G/N$ is a Lie group. By Kuiper's theorem G is weakly contractible, so the long exact sequence on homotopy groups gives an isomorphism

$$\pi_2(K) \cong \pi_1(N) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

A similar argument also shows that

$$\pi_2(\mathrm{Ad}(\mathfrak{g}/\mathfrak{n})) = \pi_2(\mathrm{Ad}\,\mathfrak{g}) = \pi_2(G/Z(G)) \cong \pi_1(Z(G)) \cong \mathbb{Z}\oplus\mathbb{Z}.$$

Indeed, this connection between integrability and π_2 is made clear by the following theorem, which gives a classification of the integrable Banach Lie algebras in terms of the period group $\Pi(\mathfrak{g})$, which is the image of a certain homomorphism $\partial : \pi_2(\mathrm{Ad}\,\mathfrak{g}) \to \mathfrak{z}$.

Theorem 11. *There is a Banach Lie group G with Lie algebra $\mathfrak{g}$ if and only if $\Pi(\mathfrak{g}) \subseteq \mathfrak{z}$ is discrete.*

In the example above the period homomorphism is given by $(m, n) \mapsto -\alpha m + n$, so its image is not discrete. In [4] there's a proof of this last theorem using essentially the same ideas we used in our proof. Moreover, we remark that in [1] a similar criterion is given for the integrability of a Lie algebroid to a Lie grupoid, and again that the construction of the Lie grupoid is a generalization of the one we described in this paper.

6. Different proofs

6.1. **Inductive proof with Levi decomposition.** With some of the structure theory for Lie algebras, in particular the Levi decomposition of a Lie algebra, we get get a very short proof of Lie's third theorem that doesn't require the full strength of Ado's theorem. This proof is presented in [10]. First, we note that if $\mathfrak{g}$ is either abelian or semi-simple then $\mathfrak{g}$ is integrable. If $\mathfrak{g}$ is abelian then $(\mathfrak{g}, +)$ is a Lie group integrating $\mathfrak{g}$. If $\mathfrak{g}$ is semi-simple then its center is trivial, so the adjoint representation is injective and therefore $\mathfrak{g} \cong \operatorname{ad} \mathfrak{g}$ is the Lie algebra of $\operatorname{Ad} \mathfrak{g}$.

We claim that if $\mathfrak{g}$ is not semi-simple neither the 1-dimensional abelian Lie algebra then we can decompose $\mathfrak{g} = \mathfrak{g}_2 \ltimes \mathfrak{g}_1$ as the semidirect product of two smaller dimension Lie algebras. If $\mathfrak{g}$ is not solvable, then the Levi decomposition

$$0 \to \operatorname{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \to 0$$

is such a decomposition since $\mathfrak{g} \neq \operatorname{Rad}(\mathfrak{g})$ because $\mathfrak{g}$ is not solvable and $\mathfrak{g} \neq \mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ because $\mathfrak{g}$ is not semi-simple. On the other hand, if $\mathfrak{g}$ is solvable then $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ and we can chose a linear subspace $\mathfrak{g}_1$ of codimension 1 in $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_1 \subseteq \mathfrak{g}$. Then automatically $\mathfrak{g}_1$ is an ideal since $[\mathfrak{g}, \mathfrak{g}_1] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_1$ and we have the following exact sequence:

$$0 \to \mathfrak{g}_1 \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_1 \cong \mathbb{R} \to 0.$$

This sequence clearly splits via the map $\mathbb{R} \to \mathfrak{g}$ defined by $1 \mapsto x$ where $x \in \mathfrak{g} \setminus \mathfrak{g}_1$.

Now we proceed by induction on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ in not semi-simple neither the 1-dimensional abelian Lie algebra we can write $\mathfrak{g} = \mathfrak{g}_2 \ltimes \mathfrak{g}_1$ where $\mathfrak{g}_2$ acts on $\mathfrak{g}_1$ by $\lambda : \mathfrak{g}_2 \to \operatorname{Der}(\mathfrak{g}_1)$. By the induction hypothesis $\mathfrak{g}_1$ and $\mathfrak{g}_2$ integrate to simply connected groups G_1 and G_2, respectively. Since $\operatorname{Der}(\mathfrak{g}_1)$ is the Lie algebra of $\operatorname{Aut}(\mathfrak{g}_1) = \operatorname{Aut}(G_1)$, by Lie's second theorem there is group homomorphism $\rho : G_2 \to \operatorname{Aut}(G_1)$ such that $\lambda = (d\rho)_e$. But then the Lie group $G = G_2 \ltimes_\rho G_1$ has Lie algebra $\mathfrak{g}_2 \ltimes_\lambda \mathfrak{g}_1 = \mathfrak{g}$ by exercise 11 in Homework 4.

6.2. **Integrating to local Lie group.** The Baker-Campbell-Hausdorff formula tells us there is a formal expression $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ written only in terms of the Lie bracket such that

$$e^X e^Y = e^{\mu(X,Y)}.$$

Explicitly, μ is given by the following formula:

$$\mu(X, Y) = Y + \sum_{n \geq 0} \frac{(-1)^n}{n+1} \sum_{r_i + s_i > 0} \frac{[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n}] X}{(1 + \sum_{i=1}^n r_i) \prod_{i=1}^n (r_i! s_i!)}$$

where the expression $[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n}]$ is defined as

$$(\operatorname{ad} X)^{r_1} (\operatorname{ad} Y)^{s_1} (\operatorname{ad} X)^{r_2} (\operatorname{ad} Y)^{s_2} \dots (\operatorname{ad} X)^{r_n} (\operatorname{ad} Y)^{s_n}$$

This formal formula converges for X, Y in a small neighborhood U of X. One can also write the Baker-Campbell-Hausdorff formula as follows: let $F(z)$ denote the formal analytic expansion of $\frac{z \log z}{z-1}$ around $z = 1$. Then

$$\mu(X, Y) = X + \int_0^1 F(\mathrm{Ad}(X)\mathrm{Ad}(tY))\, dt.$$

On the other hand, given a Lie algebra $\mathfrak{g}$ the formula μ defines a local product $\mu : U \times U \to \mathfrak{g}$ and therefore a local Lie group. Indeed, the series in μ converges and it can be shown that it really defined a Lie group structure – in particular it is locally associative, that is, $\mu(X, \mu(Y, Z)) = \mu(\mu(X, Y), Z)$ for sufficiently small $X, Y, Z \in \mathfrak{g}$, as was shown in [**?**]. This already shows Lie's first theorem, which was actually its original form: any Lie algebra is the Lie algebra of a local Lie group.

However, the problem of extending a local Lie group to a global one is a hard one, and in general it's not possible. A theorem of Mal'cev (in [6]) says that it is possible to extend a local Lie group X to a global one if and only if X obeys n-associativity for every n, that is, if for any two expressions with parentheses that multiply n elements of X give the same result whenever they are both defined. For instance 4-multiplicity implies that

$$((xy)z)w = (xy)(zw)$$

whenever both expressions are defined. The proof (of the non-trivial implication) constructs the Lie group extending X as $W(X)/\sim$ where W is the set of words on X and $\sim$ is the equivalence relation generated by

$$(x_0, \dots, x_i, x_{i+1}, \dots, x_n) \sim (x_0, \dots, x_i x_{i+1}, \dots, x_n)$$

when the product $x_i x_{i+1}$ is defined.

By Lie's third theorem, it is true that the local Lie group defined by the Baker-Campbell-Hausdorff formula respects n-associativity for every n. Indeed, a direct proof of this fact would lead to a proof of Lie's third theorem. However, this seems highly non-trivial, as the proof of 3-associativity itself is already quite hard.

On the other hand, in [9] it's proven that, although we may be unable to extend a local Lie group to a global one, we can extend some (local) cover of the local Lie group. Doing this with the local Lie group given by the Baker-Campbell-Hausdorff formula constructs a Lie group integrating $\mathfrak{g}$.

6.3. **Integration of the adjoint extension.** We will now sketch the idea of another proof, based on the ideas of van Est on how to integrate an abelia extension applied to the adjoint extension of a Lie algebra. The full details for such a proof can be found in [11].

Consider the adjoint extension of the Lie algebra $\mathfrak{g}$:

$$0 \to \mathfrak{z} \to \mathfrak{g} \to \mathrm{ad}\, \mathfrak{g} \to 0.$$

Both $\mathfrak{z}$ and $\mathfrak{h} \equiv \mathrm{ad}\, \mathfrak{g}$ are integrable to $Z = (\mathfrak{z}, +)$ and $H \equiv \widetilde{Ad(\mathfrak{g})}$, respectively. Hence, we would like to find integrate $\mathfrak{g}$ to a simply-connected Lie group G which is an extension of H by Z, i.e. that fits in a short exact sequence

$$0 \to Z \to G \to H \to 0.$$

Recall that the extensions of $\mathfrak{h}$ by $\mathfrak{z}$ are classified by an action $\mathfrak{h} \to \mathrm{Aut}(\mathfrak{z})$ and an element of the Lie algebra cohomology $H^2_{alg}(\mathfrak{h}, \mathfrak{z})$. On the other hand, extensions of H by Z are classified by an action $H \to \mathrm{Aut}(Z)$ and by an element of the cohomology $H^2_{gr}(H, Z)$ where H^2_{gr} is a certain cohomology theory for Lie groups – note that in [11] what we're calling H^2_{gr} is $H^2_{gr,es}$, while there H^2_{gr} denotes the "discrete group cohomology". By Lie's second theorem one can easily integrate the Lie algebra action $\mathfrak{h} \to \mathrm{Aut}(\mathfrak{z})$ to a Lie group action $H \to \mathrm{Aut}(Z)$. Hence, it would be enough to understand how to integrate the element of the Lie algebra cohomology $H^2_{alg}(\mathfrak{h}, \mathfrak{z})$ to an element of $H^2_{gr}(H, Z)$, and this is possible (and this is possible as long as H is simply-connected).

Indeed there is a map $\Delta : H^2_{gr}(H, Z) \to H^2_{alg}(\mathfrak{h}, \mathfrak{z})$, which we can regard as differentiation of cochains, such that if $[\varphi] \in H^2_{gr}(H, Z)$ represents an extension $\pi : G \to H$ then $\Delta[\varphi] \in H^2_{alg}(\mathfrak{h}, \mathfrak{z})$ represents the extension $(d\pi)_e : \mathfrak{g} \to \mathfrak{h}$. It was proven by van Est in [12] that Δ is an isomorphism if H is simply connected – indeed he proved that, more generally, the differentiation map $H^\ell_{gr}(H, Z) \to H^\ell_{alg}(\mathfrak{h}, \mathfrak{z})$ is an isomorphism if H is ℓ-connected. This is the case for $H = \widetilde{\mathrm{Ad}(\mathfrak{g})}$, showing that we can find an extension $G \to H$ corresponding via the isomorphism Δ to the extension $\mathfrak{g} \to \mathfrak{h}$; the Lie group G found in this way integrates the Lie algebra $\mathfrak{g}$.

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