A GEOMETRIC CRITERION FOR THE BOUNDEDNESS OF CHARACTERISTIC CLASSES

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Abstract. We show that for a connected Lie group G, the linearity of its radical $\sqrt{G}$ (that is of its biggest connected normal solvable subgroup), is a necessary and sufficient condition for the boundedness of all Borel cohomology classes of G with integer coefficients, and that the linearity of $\sqrt{G}$ is also equivalent to a large-scale geometric property of G (involving distortion).

1. Introduction

Let G be a Lie group and let G^δ be the same group endowed with the discrete topology. The identity map $G^\delta \to G$ induces a map between classifying spaces $BG^\delta \to BG$. Gromov proved in 1982 that if G is the real Lie group associated to a linear algebraic group defined over $\mathbb{R}$, then the range of the natural map

$$H^*(BG, \mathbb{R}) \to H^*(BG^\delta, \mathbb{R}),$$

between the singular cohomology groups, consists of bounded classes. In other words, following the terminology introduced in [17, Section 1.3, p. 23]: *primary characteristic classes of a linear algebraic group over $\mathbb{R}$ are bounded.* In view of a result proved by Wigner in 1973 [36, Theorem 4, p. 93], Gromov's result may be reformulated as follow: *all Borel cohomology classes with integer coefficients of a linear algebraic group over $\mathbb{R}$ are bounded.*

It turns out that *neither the algebraicity nor the linearity of the group are necessary conditions in Gromov's result*: relying on Gromov's result and its proof by Bucher-Karlsson, we show in Section 5 that *the linearity of the radical of G is a sufficient condition for the boundedness of all primary characteristic classes of G.* Recall that there are connected Lie groups which are linear but not algebraic. Also there are connected Lie groups with linear radical but which do not cover any linear Lie group. (The quotient of the product of the three dimensional Heisenberg group with the universal cover of $SL(2, \mathbb{R})$ by a central diagonal infinite cyclic subgroup is such an example.) The linearity of the radical of a connected Lie group G is

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Date: September 16, 2010.

2000 *Mathematics Subject Classification.* Primary: 57T10, 55R40; Secondary: 20F65, 53C23.

Key words and phrases. Bounded cohomology, Lie groups, classifying spaces, Borel cocycles, word metrics.

I. Chatterji is partially supported by the NSF grant No. 0644613 and the ANR grant JC-318197 QuantiT, Ch. Pittet is partially supported by the CNRS. L. Saloff-Coste is partially supported by NSF grant DMS-0603886. The authors are grateful to the following institutions where part of this work was achieved: Aix-Marseille I University, the CIRM at Luminy, Cornell University, the Erwin Schrödinger Institute, the Institut Henri-Poincaré, the Laboratoire Poncelet at the Independent University of Moscow, Ohio State University.

still too strong an hypothesis for the boundedness of all the primary characteristic classes of G, see Remark 5.2 at the end of the paper.

Our main result is that *the linearity of the radical of a connected Lie group G, is a necessary and sufficient condition for the boundedness of all Borel cohomology classes of G with integer coefficients, and that the linearity of the radical is also equivalent to a large-scale geometric property of G (involving distortion).*

We recall the definitions needed to state our results. A *Borel n-cochain* on G with values in $\mathbb{Z}$ is a map

$$c : G^n \to \mathbb{Z},$$

such that the inverse image of any subset of $\mathbb{Z}$ is a Borel set (i.e. an element of the σ-algebra generated by the open subsets of G^n endowed with its product topology). The usual differentials send Borel cochains to Borel coboundaries. We denote by

$$H^*_B(G, \mathbb{Z})$$

the corresponding cohomology group (see Section 3 below for details and references).

Recall that a class in $H^*_B(G, \mathbb{Z})$ is *bounded* if it admits a representative Borel cocycle whose image is finite.

We refer the reader to Subsection 2.2, Definition 2.14 below for the definition of an *undistorted* subgroup.

Theorem 1.1. *Let G be a connected Lie group. The following conditions are equivalent.*

(1) The radical of G is linear.
(2) Each Borel cohomology class of G with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.
(3) The natural inclusion $\pi_1(G) \to \tilde{G}$ of the fundamental group of G into the universal cover of G is undistorted.

The implication (1)⇒(2) is proved in two steps. We first show that (1) implies that 2-dimensional integral Borel cohomology classes are bounded, using an interpretation of central extensions of topological groups in terms of Borel cocycles, due to Moore [30]. We then show, relying on the boundedness result on real primary characteristic classes of flat bundles with structure group a real linear algebraic Lie group, due to Gromov [17, Section 1.3, p. 23] and Bucher-Karlsson [9], how to pass to cohomology classes of degree larger than 2. The implication (2) ⇒ (3) relies on properties of distorted discrete central subgroups of Lie groups and the equivalence of (1) and (3) is dealt with using Lie group and Lie algebra techniques.

In Section 2 we recall some facts on Lie groups and prove the results concerning distortion of subgroups used later on. Borel cohomology of topological groups, which is based on Borel (measurable) cochains, is discussed in Section 3. In Section 4 we present the proof of Theorem 1.1 and mention in Corollary 4.4 two more equivalent conditions to the ones of Theorem 1.1. In Section 5 we use Wigner's result mentioned above to explain the relationship of Theorem 1.1 with Gromov's Theorem.

Theorem 1.1 has roots in [11, Proposition 5.5 and Lemma 6.3], where some of the authors of the present paper needed to work with Borel cocycles associated to distorted and undistorted central extensions.

Acknowledgments. We thank Marc Burger, Alessandra Iozzi, Nicolas Monod, Antony Wassermann, and Dave Witte Morris, for useful discussions and comments.

We are grateful to Kenneth Brown for mentioning to us [30, Theorem 10]. We are indebted to Yves de Cornulier who explained to us the geometrical meaning (see Remark 2.23) of the condition $\pi_1([\sqrt{G}, \sqrt{G}]) = 0$.

2. On the geometry and topology of Lie groups

2.1. **Preliminaries and facts on Lie groups.** In this subsection we discuss general facts about the topology of Lie groups and their algebraic structure. We also recall linearity criteria for Lie groups (Theorem 2.1 and Proposition 2.9).

To fix notation, we recall some basic facts on Lie groups; a good reference is [32]. Let G be a connected (real) Lie group. Then G admits a Levi decomposition $G = \sqrt{G} \cdot L(G)$ with $\sqrt{G}$ the radical of G (the biggest connected solvable normal subgroup) and $L(G)$ a Levi subgroup (a maximal connected semi-simple subgroup; the subgroups $\sqrt{G}$ and $L(G)$ are analytic (a subgroup $H < G$ is called *analytic*, if H is a connected Lie group and the inclusion $H \to G$ is an immersion of manifolds). The intersection $\sqrt{G} \cap L(G)$ is a totally disconnected subgroup of G, which is discrete in $L(G)$ but in general not discrete in $\sqrt{G}$. In case G is simply-connected, $\sqrt{G} \cap L(G) = \{e\}$; hence G is the semi-direct product $G = \sqrt{G} \rtimes L(G)$. A group G is called *linear*, if it admits a faithful representation $G \to GL(n, \mathbb{R})$. In case G is a connected Lie group, there is a closed normal subgroup $\Lambda(G)$, the *linearizer* of G, such that $G/\Lambda(G)$ is linear and such that any Lie homomorphism $G \to H$ with H linear factors through $G/\Lambda(G)$ (for the structure of the linearizer see [23]). If G is connected and semi-simple, $\Lambda(G)$ is a central discrete subgroup and any quotient Lie group of $G/\Lambda(G)$ is linear too [32, Ch. 5, §3, Theorem 8]. In case G is connected and solvable, $\Lambda(G)$ is a central torus; a quotient of a linear solvable Lie group need not be linear. Concerning the linearity of Lie groups, the following theorem is basic.

Theorem 2.1. (Malcev [26].) *A connected Lie group G is linear if and only if its radical $\sqrt{G}$ and Levi subgroup $L(G)$ are.*

In the sequel we will also deal with not necessarily connected Lie groups G; we will write G^0 for the connected component of G. By the radical $\sqrt{G}$ of G we still mean a maximal connected normal solvable subgroup of G, thus $\sqrt{G} = \sqrt{G^0}$. A group G is called *virtually connected*, if G/G^0 is a finite group. In case G_a is a connected linear algebraic group defined over $\mathbb{R}$, its Lie group of $\mathbb{R}$-points $G_a(\mathbb{R})$ is virtually connected. Every connected linear reductive Lie group G is, as a Lie group, isomorphic to $G_a(\mathbb{R})^0$ for some connected linear real algebraic group G_a (see [25]).

The following simple observation is used later.

Lemma 2.2. *Let G be a contractible Lie group and $H < G$ an analytic subgroup. Then H is contractible too.*

Proof. Recall that any connected Lie group is homotopy equivalent to a maximal compact subgroup. Let K be a maximal compact subgroup of H. Then K is contained in a maximal compact subgroup of G, which is $\{e\}$. Thus $K = \{e\}$ and therefore H is contractible. □

Since a maximal compact subgroup of a connected solvable Lie group S is a torus, S is contractible if and only if S is simply-connected. It follows that every connected analytic subgroup of a simply-connected solvable Lie group is contractible.

Lemma 2.3. *Let N be a connected closed normal subgroup of a connected Lie group G. Then the inclusion $N \hookrightarrow G$ induces a short exact sequence $0 \to \pi_1(N) \to \pi_1(G) \to \pi_1(G/N) \to 0$.*

Proof. The long exact sequence in homotopy reads

$$\cdots \to \pi_2(G/N) \to \pi_1(N) \to \pi_1(G) \to \pi_1(G/N) \to 0\ .$$

Because G/N is a Lie group, $\pi_2(G/N) = 0$ (see [7]), and the result follows. □

Remark 2.4. *Since $\mathbb{R}$ is divisible,* $\mathrm{Hom}(-, \mathbb{R})$ *is an exact contravariant functor on abelian groups. As the fundamental group of a Lie group is abelian, Lemma 2.3 shows that the inclusion $N \hookrightarrow G$ induces a surjection*

$$\mathrm{Hom}(\pi_1(G), \mathbb{R}) \twoheadrightarrow \mathrm{Hom}(\pi_1(N), \mathbb{R}).$$

Lemma 2.5. *Let N be a closed connected normal subgroup of a connected Lie group G. There are two exact sequences with commutative squares*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma \cap \tilde{N} & \longrightarrow & \tilde{N} & \longrightarrow & N & \longrightarrow & 1,
\end{array}$$

were $\tilde{G}$ and $\tilde{N}$ are the respective universal covers of G and N, whereas Γ and $\Gamma \cap \tilde{N}$ are central discrete subgroups, isomorphic to the respective fundamental groups of G and N, and the vertical arrows are injections of closed normal subgroups.

Proof. Let $p : \tilde{G} \to G$ be the universal cover of G. The connected component C of the identity in $p^{-1}(N)$ is a closed connected normal subgroup of $\tilde{G}$ and the restriction of p to C is a cover of N. According to Lemma 2.3 the group C is simply-connected. Hence $C = \tilde{N}$, the universal cover of N. □

Lemma 2.6. *Let G be a connected nilpotent Lie group. Let $\phi : \mathbb{R} \to G$ be a one-parameter subgroup. If $\phi(t_0)$ is central for some $t_0 \neq 0$ then $\phi(t)$ is central for all t.*

Proof. As the universal cover of a Lie group G is an extension of G with discrete central kernel, an easy argument shows that it is enough to prove the lemma when the nilpotent group G is simply-connected. In this case, the exponential map

$$\exp : \mathfrak{g} \to G$$

is a diffeomorphism. In exponential coordinates the Lie group multiplication is given by the Campbell-Hausdorff formula

$$xy = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + ...$$

for all $x, y \in \mathfrak{g}$. As the Lie algebra $\mathfrak{g}$ is nilpotent, the above expression has finitely many terms. Hence if $g \in G$ is given, the equation $g\phi(t)g^{-1}\phi(t)^{-1} = e$ is given by $d = \dim(\mathfrak{g})$ polynomial equations $P_i(t) = 0$, $1 \leq i \leq d$. By hypothesis, for each i, we have $P_i(nt_0) = 0$ for all integer n. Hence the equality holds for all $t \in \mathbb{R}$. □

The following well-known fact is an immediate consequence.

Lemma 2.7. *A connected nilpotent Lie group N has a central torus as its unique maximal compact subgroup.*

Proof. Since a connected compact subgroup of N is a torus, and since the union of all 1-dimensional tori in a given torus T is a dense subset of T, it suffices to prove that every 1-dimensional torus $S^1 < N$ is central. Choose a surjective homomorphism $\phi : \mathbb{R} \to S^1 < N$ and $t_0 \in \mathbb{R} \setminus \{0\}$ with $\phi(t_0) = e$. Then the results follows by applying Lemma 2.6. $\square$

Recall that the *nilradical* of a Lie group is the largest connected, normal, nilpotent subgroup.

Lemma 2.8. *Let K be a compact subgroup of a connected solvable Lie group G. Let T_N be the maximal compact subgroup of the nilradical N of G. If $k \in [G,G] \cap K$ then there exists a one-parameter subgroup in $T_N \cap [G,G]$ passing through k.*

Proof. Let $k \in [G,G] \cap K$ be a non-trivial element. Let $\mathfrak{D}$ be the Lie algebra of the (not necessarily closed) analytic subgroup $[G,G]$. Since G is solvable, $[G,G]$ is nilpotent and the exponential map

$$\exp : \mathfrak{D} \to [G,G]$$

is surjective (see [22]). Let $X \in \mathfrak{D}$ be such that $\exp(X) = k$ and denote by $\phi : \mathbb{R} \to [G,G]$ the corresponding one-parameter subgroup $\phi(t) = \exp(tX)$. Since $[G,G]$ is a normal nilpotent connected subgroup of G, it is contained in the nilradical N of G. Let T_N be the maximal compact subgroup of N. We hence have that $[G,G] \cap K \subseteq T_N$. Since $k = \exp(X) \in T_N$, it implies that $\phi(\mathbb{Z}) \subseteq T_N$. The 1-parameter subgroup $\phi(\mathbb{R})$, being at bounded distance from $\phi(\mathbb{Z})$ (for any left-invariant Riemannian metric on G), is bounded and hence the closure of $\phi(\mathbb{R})$ in N is a compact subgroup. We conclude that $\phi(\mathbb{R}) \subseteq T_N \cap [G,G]$. $\square$

Proposition 2.9. *(Malcev, Gotô) Let G be a connected solvable Lie group. The following are equivalent:*

(1) The group G is linear.
(2) The group $\pi_1\left(\overline{[G,G]}\right)$ is trivial.
(3) There is a maximal compact subgroup $K < G$ and a closed 1-connected normal subgroup $H < G$ such that $G = HK$.

The equivalence of (1) and (2) was proved by Gotô, see [16, Theorem 5], and that of (1) and (3) is due to Malcev [26].

Remark 2.10. *In case the group G is nilpotent, the equivalent conditions of Proposition 2.9 are also equivalent to the following one:* the group is a direct product $G = T \times N$ where T is the unique maximal compact subgroup of G and where N is nilpotent and contractible. *This follows from Lemma 2.7.*

Remark 2.11. *The equivalent conditions of Proposition 2.9 are not equivalent to the group $\pi_1([G,G])$ being trivial. Indeed, the following is an example of a connected solvable Lie group G with $[G,G]$ simply-connected but $\overline{[G,G]}$ not. Let $\mathbf{H}$ be the 3-dimensional Heisenberg group and consider $\mathbf{H} \times S^1$. Its center is $\mathbb{R} \times S^1$. Take the discrete central subgroup $\mathbb{Z}$ generated by $(1,t)$ with 1 generating $\mathbb{Z}$ in $\mathbb{R}$, and t of infinite order in S^1; this central subgroup of $\mathbf{H} \times S^1$ is discrete. Let us define $G := (\mathbf{H} \times S^1)/\mathbb{Z}$. It is a nilpotent connected Lie group with $[G,G]$ homeomorphic to $\mathbb{R}$, embedded in the maximal torus $S^1 \times S^1$ of G in a dense way. It follows that $\pi_1([G,G])$ is trivial but $\pi_1\left(\overline{[G,G]}\right) = \mathbb{Z} \times \mathbb{Z}$. Thus G is not linear (its linearizer is*

$\overline{[G,G]}$, a central 2-torus). We would like to warn the reader that in several places in the literature one finds an incorrect statement saying that a connected solvable Lie group G is linear if and only if $[G,G]$ is simply-connected (for instance, see [35, Ch. 2, Theorem 7.1]*); the correct statement is that G is linear if and only if the closure $\overline{[G,G]} < G$ is simply-connected.*

For our purpose the following variation of Proposition 2.9 will be useful.

Lemma 2.12. *Let G be a connected solvable Lie group. The following are equivalent:*

(1) The group G is linear.
(2) For every maximal compact subgroup $K < G$, $\overline{[G,G]} \cap K = \{e\}$.
(3) For every maximal compact subgroup $K < G$, $[G,G] \cap K = \{e\}$.

Proof. (1)⇒(2)⇒(3): If G is linear, $G = HK$ with $K < G$ maximal compact and H simply-connected, normal and closed (cf. Proposition 2.9); we can choose here any maximal compact subgroup of $K < G$, because these are all conjugate. This implies that $H \cap K = \{e\}$ so that $G/H \cong K$ is commutative, because K is a torus. Thus $\overline{[G,G]} < H$ and (2) and then (3) follow. It remains to show that (3) implies (1). If G is not linear, the maximal compact subgroup T of $\overline{[G,G]}$ is a non-trivial torus because $\pi_1\left(\overline{[G,G]}\right) \neq \{e\}$. The assumption (3) implies that $T \cap [G,G] = \{e\}$. Because $\overline{[G,G]}$ is nilpotent, T is a central subgroup of $\overline{[G,G]}$ (Lemma 2.7). Therefore,

$$\phi : T \times [G,G] \to \overline{[G,G]}, \quad (t,x) \mapsto tx$$

maps isomorphically onto the dense subgroup $T[G,G] < \overline{[G,G]}$. Under the projection $\overline{[G,G]} \to \overline{[G,G]}/T$, $[G,G]$ maps isomorphically onto a dense analytic subgroup $U < \overline{[G,G]}/T$. Since $\overline{[G,G]}/T$ is simply-connected and U is normal, U is closed and therefore equal to $\overline{[G,G]}/T$. It follows that the pre-image $T[G,G]$ is equal to $\overline{[G,G]}$ and we conclude that ϕ is an isomorphism of Lie groups. This implies that $[G,G]$ is dense in $T \times [G,G]$, which contradicts the assumption that T is non-trivial. □

Lemma 2.13. *Let G be a connected Lie group, $\sqrt{G}$ its radical and $Q = G/\sqrt{G}$ simply-connected. The short exact sequence*

$$1 \to \sqrt{G} \to G \to Q \to 1$$

splits, and therefore we have a semi-direct product, $G = \sqrt{G} \rtimes Q$.

Proof. This is a classical result in the case G itself is simply-connected (see [22]). The proof in our case is reduced to the simply-connected case by considering the following two exact sequences with commutative squares,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \sqrt{\tilde{G}} = p^{-1}(\sqrt{G})^0 & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{G}/\sqrt{\tilde{G}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow p & & \downarrow \phi & & \\
1 & \longrightarrow & \sqrt{G} & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1,
\end{array}$$

where $\tilde{G}$ is the universal cover of G. The map ϕ is an isomorphism because it is a connected covering of a simply-connected space. Thus, if σ is a splitting for $\tilde{G} \to \tilde{G}/\sqrt{\tilde{G}}$, then $p \circ \sigma \circ \phi^{-1}$ is a splitting for $G \to Q$. □

2.2. **Distorted and undistorted central extensions.** In this subsection, we first recall some basic facts about the distortion of subgroups. We establish an algebraic criterion to decide when certain central subgroups of a simply-connected solvable Lie group are distorted (Proposition 2.18). Finally, we show that if the radical of a connected Lie group G is not linear, then the fundamental group of G is distorted in the universal cover of G (Proposition 2.21).

Definition 2.14. *Let A and E be two locally compact, compactly generated groups. We denote by L_S, resp. L_U, the word length associated to a symmetric relatively compact generating set S of A, resp. U of E. Assume that A is a subgroup of E. We say that A is* undistorted *in E if the identity map is a quasi-isometry between (A, L_S) and $(A, L_U|_A)$. Otherwise we say that the subgroup A is* distorted *in E. If A is the kernel of a continuous homomorphism $E \twoheadrightarrow G$, we call the extension*

$$1 \to A \to E \to G \to 1$$

undistorted*, resp.* distorted*, if A is undistorted, resp. distorted in E.*

Remarks 2.15. *(1) Under the hypothesis of the above definition, Gromov defines in* [18, Chapter 3]*, the distortion function as*

$$\textsc{Disto}(r) := \frac{\operatorname{diam}_A(A \cap B_E(r))}{r}, \forall r > 0.$$

One checks that A is undistorted exactly when the function Disto *is bounded.*
(2) We say that the distortion of A is at least linear *if there exist $a > 0$ and $R > 0$ such that for all $r \geq R$, we have*

$$ar \leq \textsc{Disto}(r).$$

We say that the distortion is sub-linear *if it is not at least linear.*
(3) Having a bounded, or unbounded, or sub-linear, etc., distortion function, is a well defined property of the couple $A < E$, i.e. does not depend on the choice of the relatively compact symmetric generating sets.

Lemma 2.16. *Let $p : G \to Q$ be a continuous homomorphism between locally compact, compactly generated groups. Let $H < G$ be a compactly generated subgroup of G and assume that $H \cap \ker(p) = \{e\}$. If H is distorted in G then so is $p(H)$ in Q.*

Proof. Let S be a compact symmetric generating set of H. Then $p(S)$ is a compact symmetric generating set of $p(H)$. Let $h \in H$. As p is a homomorphism, $L_{p(S)}(p(h)) \leq L_S(h)$. As $H \cap \ker(p) = \{e\}$, we also have $L_{p(S)}(p(h)) \geq L_S(h)$. The proof now follows because a homomorphism is C-Lipschitz with respect to word metrics (for $C \geq 1$ a constant depending on p and on generating sets) and because the projection of a path between the identity in G and $h \in H$ is still a path between the identity in Q and $p(h) \in p(H)$. □

Proposition 2.17. (Gromov [18, 3B2].) *Let N be a simply-connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $x \in [\mathfrak{n}, \mathfrak{n}] \setminus \{0\}$. Then the one-parameter subgroup $t \mapsto \exp(tx)$, is at least linearly distorted in N.*

The statement in [18, 3B2] which implies Proposition 2.17 is not proved. For a proof of Proposition 2.17, we refer the reader to [34, Prop. 4.1] or [33]. A more conceptual proof, in the spirit of [18], follows from the existence of a homothety, relative to a Carnot-Caratheodory metric on N, in the case N is graded. The general case is reduced to the graded case using [6, Theorem 1.3].

Proposition 2.18. *Let $1 \to N \to S \to A \to 1$ be a short exact sequence of connected Lie groups with N nilpotent S simply-connected and A abelian. Let $Z < N$ be a connected analytic subgroup, which is central in S. Denote by $\mathfrak{z}$, resp. $\mathfrak{s}$ the Lie algebras of Z, resp. S. Then either $\mathfrak{z}$ is a direct factor in $\mathfrak{s}$, or there exists a one-parameter subgroup in Z which is distorted in S.*

Proof. Let $T \in \mathfrak{s}$ be a regular element, and let $\mathfrak{n}(T,\mathfrak{s})$ be the associated Cartan sub-algebra [12, Chapitre VI, Paragraphe 4, 2, Proposition 9, p. 387]. The sub-algebra $\mathfrak{n}(T,\mathfrak{s})$ is the big kernel of $ad(T)$; that is, there is an integer $N \in \mathbb{N}$, big enough such that $\mathfrak{n}(T,\mathfrak{s}) = \ker\left(ad^N(T)\right) = \ker\left(ad^{N+1}(T)\right)$. Let $\mathfrak{i}(T,\mathfrak{s})$ be the small image of $ad(T)$; that is, there is an integer $M \in \mathbb{N}$, big enough such that $\mathfrak{i}(T,\mathfrak{s}) = ad^M(T)(\mathfrak{s}) = ad^{M+1}(T)(\mathfrak{s})$. Hence $\mathfrak{s}$ decomposes as a direct sum of $ad(T)$-invariant subspaces,

$$\mathfrak{s} = \mathfrak{n}(T,\mathfrak{s}) \oplus \mathfrak{i}(T,\mathfrak{s}).$$

The derivation $ad(T)$ has a semi-simple part $ad_s(T)$, that gives $\mathfrak{n}(T,\mathfrak{s}) = \ker\left(ad_s(T)\right)$, and $\mathfrak{i}(T,\mathfrak{s}) = ad_s(T)(\mathfrak{s})$, see [12, Chapitre VI, Paragraphe 4, 2, Proposition 8, p. 385]. It follows that

$$[\mathfrak{n}(T,\mathfrak{s}), \mathfrak{i}(T,\mathfrak{s})] \subseteq \mathfrak{i}(T,\mathfrak{s}),$$

because if $X \in \mathfrak{n}(T,\mathfrak{s})$ and $Y \in \mathfrak{i}(T,\mathfrak{s})$, we can choose $\tilde{Y} \in \mathfrak{s}$ such that $ad(T)(\tilde{Y}) = Y$, hence

$$[X,Y] = [X, ad_s(T)(\tilde{Y})] + [ad_s(T)(X), \tilde{Y}] = ad_s(T)([X,\tilde{Y}]).$$

We note that the Cartan subgroup $C < S$ associated to $\mathfrak{n}(T,\mathfrak{s})$ is, by Lemma 2.2, a simply-connected nilpotent group, because S is a simply-connected solvable Lie group and therefore contractible. If the intersection $\mathfrak{z} \cap [\mathfrak{n}(T,\mathfrak{s}), \mathfrak{n}(T,\mathfrak{s})]$ is non-trivial, then according to Proposition 2.17, there is a one-parameter subgroup in Z which is distorted in the Cartan subgroup $C < S$. This obviously implies that Z is distorted in S as well, and the proof is finished in this case.

So we can assume that $\mathfrak{z} \cap [\mathfrak{n}(T,\mathfrak{s}), \mathfrak{n}(T,\mathfrak{s})] = \{0\}$. We notice that as the sub-algebra $\mathfrak{z}$ is central in $\mathfrak{s}$, it is contained in $\mathfrak{n}(T,\mathfrak{s})$. Hence we can choose a complement V for $\mathfrak{z} \oplus [\mathfrak{n}(T,\mathfrak{s}), \mathfrak{n}(T,\mathfrak{s})]$ in $\mathfrak{n}(T,\mathfrak{s})$. Obviously $\mathfrak{m} = [\mathfrak{n}(T,\mathfrak{s}), \mathfrak{n}(T,\mathfrak{s})] \oplus V$ is a sub-algebra of $\mathfrak{n}(T,\mathfrak{s})$ and hence $\mathfrak{n}(T,\mathfrak{s}) = \mathfrak{z} \times \mathfrak{m}$. Let $\mathfrak{u} = [\mathfrak{n},\mathfrak{n}] \cap \mathfrak{n}(T,\mathfrak{s})$, where $\mathfrak{n}$ denotes the Lie algebra of N.

We can assume that $\mathfrak{z} \cap \mathfrak{u} = \{0\}$. Because otherwise, according to Proposition 2.17, there is a one-parameter subgroup in Z which is distorted in the analytic subgroup of S associated to $\mathfrak{n}$, hence in S as well. Let $q : \mathfrak{z} \times \mathfrak{m} \to \mathfrak{m}$ be the projection onto the second factor. Let V' be a complement for $\mathfrak{u} \cap \mathfrak{m}$ in $\mathfrak{u}$, that is $(\mathfrak{u} \cap \mathfrak{m}) \oplus V' = \mathfrak{u}$. If $q(V') \cap ([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) \neq \{0\}$, we claim that there is a one-parameter subgroup in Z which is distorted in S. To see why, let $v \in V'$ such that $q(v) = x + y \neq 0$, with $x \in [\mathfrak{m},\mathfrak{m}]$ and $y \in \mathfrak{u} \cap \mathfrak{m}$. The element $z = v - (x+y)$ belongs to $\mathfrak{z}$. Notice that $z \neq 0$, because otherwise we would have $v = x + y \in V' \cap (\mathfrak{u} \cap \mathfrak{m}) = \{0\}$. Similarly $x \neq 0$, because otherwise $z = v - y \in \mathfrak{u} \cap \mathfrak{z} = \{0\}$. Since $x \in [\mathfrak{m},\mathfrak{m}]$, the one-parameter subgroup $t \mapsto \exp(tx)$ is distorted in S, according to Proposition 2.17. Since $v - y \in \mathfrak{u} \subseteq [\mathfrak{n},\mathfrak{n}]$, the one-parameter subgroup $t \mapsto \exp(t(v-y)x)$ is distorted in S, according to Proposition 2.17. The sub-algebra of $\mathfrak{s}$ spanned by x and z is isomorphic to $\mathbb{R}^2$. Hence, as $z = (-x) + (v-y)$, the one-parameter subgroup

$t \mapsto \exp(tz)$ is also distorted in S. (The geometric picture is the following. To reach the element $\exp(n^2 z)$ where n is large, we start from the identity in S and we first reach with a path in $\exp([\mathfrak{m},\mathfrak{m}])$, of length linear in n, the point $\exp(n^2(-x))$. Then we follow a path between $\exp(n^2(-x))$ and $\exp(n^2 z)$, obtained as the left-translated in S by $\exp(n^2(-x))$ of a path in $\exp([\mathfrak{n},\mathfrak{n}])$ between the identity and $\exp(n^2(v-y))$ and of length linear in n.)

Hence we assume that $q(V') \cap ([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) = \{0\}$, and we will show that $\mathfrak{z}$ is a direct factor in $\mathfrak{s}$. We consider a complement W for $q(V') \oplus ([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m})$ in $\mathfrak{m}$. Hence

$$\mathfrak{m} = ([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) \oplus q(V') \oplus W.$$

We define the sum of subspaces,

$$\tilde{\mathfrak{m}} = ([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + V' + W.$$

We claim that $\mathfrak{z} \times \tilde{\mathfrak{m}} \cong \mathfrak{n}(T,\mathfrak{s})$. The inclusion $\mathfrak{z} + \tilde{\mathfrak{m}} \subseteq \mathfrak{n}(T,\mathfrak{s})$ is obvious. The opposite inclusion is true because if $x \in \mathfrak{n}(T,\mathfrak{s})$, then $x = z + y$ with $z \in \mathfrak{z}$ and $y \in \mathfrak{m}$, and we can write $y = y_1 + y_2 + y_3$ with $y_1 \in [\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}$, $y_2 \in q(V')$, and $y_3 \in W$. Let $v \in V'$ such that $q(v) = y_2$. Define $z' = v - y_2 \in \mathfrak{z}$. Hence $x = z + y = (z - z') + y_1 + v + y_3$, with $z - z' \in \mathfrak{z}$, $y_1 \in [\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}$, $v \in V'$, and $y_3 \in W$. The sum is direct because,

$$\begin{aligned} \dim(\tilde{\mathfrak{m}}) &\leq \dim([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + \dim(V') + \dim(W) \\ &= \dim([\mathfrak{m},\mathfrak{m}] + \mathfrak{u} \cap \mathfrak{m}) + \dim(q(V')) + \dim(W) = \dim(\mathfrak{m}). \end{aligned}$$

As the subspace $\tilde{\mathfrak{m}}$ of $\mathfrak{n}(T,\mathfrak{s})$ is a sub-algebra (because it contains $[\mathfrak{n}(T,\mathfrak{s}),\mathfrak{n}(T,\mathfrak{s})] = [\mathfrak{m},\mathfrak{m}]$), and as $\mathfrak{z}$ is central, we obtain a direct product as claimed. What we gained in replacing $\mathfrak{m}$ with $\tilde{\mathfrak{m}}$, is that the latest contains $\mathfrak{u}$ because $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{m}) \oplus V' \subseteq \tilde{\mathfrak{m}}$. This will be crucial in finishing the proof.

We have:

$$\mathfrak{s} = \mathfrak{n}(T,\mathfrak{s}) \oplus \mathfrak{i}(T,\mathfrak{s}) = \mathfrak{z} \oplus \tilde{\mathfrak{m}} \oplus \mathfrak{i}(T,\mathfrak{s}).$$

The proof will be finish if we show that $\tilde{\mathfrak{m}} \oplus \mathfrak{i}(T,\mathfrak{s})$ is a sub-algebra of $\mathfrak{s}$. Let $x, x' \in \tilde{\mathfrak{m}}$ and $y, y' \in \mathfrak{i}(T,\mathfrak{s})$. We have,

$$[x + y, x' + y'] = \underbrace{[x, x']}_{\in \tilde{\mathfrak{m}}} + \underbrace{[x, y'] + [y, x']}_{\in \mathfrak{i}(T,\mathfrak{s})} + [y, y'].$$

Hence we have to show that $[y, y'] \in \tilde{\mathfrak{m}} \oplus \mathfrak{i}(T,\mathfrak{s})$. As $\mathfrak{i}(T,\mathfrak{s}) \subseteq \ker(p) = \mathfrak{n}$, we have $[\mathfrak{i}(T,\mathfrak{s}),\mathfrak{i}(T,\mathfrak{s})] \subseteq [\mathfrak{n},\mathfrak{n}]$. As $\mathfrak{n}$ is an ideal of $\mathfrak{s}$, it is preserved by $ad(T)$. Hence, as $ad(T)$ is a derivation it also preserves $[\mathfrak{n},\mathfrak{n}]$. Let $\mathfrak{n}(T,[\mathfrak{n},\mathfrak{n}])$, resp. $\mathfrak{i}(T,[\mathfrak{n},\mathfrak{n}])$, be the big kernel, resp. the small image, of the restriction of $ad(T)$ to $[\mathfrak{n},\mathfrak{n}]$. We have

$$[\mathfrak{n},\mathfrak{n}] = \mathfrak{n}(T,[\mathfrak{n},\mathfrak{n}]) \oplus \mathfrak{i}(T,[\mathfrak{n},\mathfrak{n}]).$$

Now we can conclude because $\mathfrak{n}(T,[\mathfrak{n},\mathfrak{n}]) \subseteq \mathfrak{n}(T,\mathfrak{s}) \cap [\mathfrak{n},\mathfrak{n}] = \mathfrak{u} \subseteq \tilde{\mathfrak{m}}$ and $\mathfrak{i}(T,[\mathfrak{n},\mathfrak{n}]) \subseteq \mathfrak{i}(T,\mathfrak{s})$. ☐

Remark 2.19. *In fact, the proof of Proposition 2.18 shows that if the distortion function of the central one-parameter subgroup under consideration is not bounded, then it grows at least linearly.*

Lemma 2.20. *Let V be a simply-connected abelian Lie group with a left-invariant Riemannian metric and let Γ be a lattice of V. Let $t \mapsto \exp(tX)$ be a one-parameter subgroup of V which projects to a dense subgroup of V/Γ. Let $\Gamma = \mathbb{Z} \oplus A$ be a splitting of Γ and let z be a generator of the $\mathbb{Z}$-factor. Then there are constants*

$C > 1$, $0 < \alpha < 1 < \beta$, such that for each $n \in \mathbb{Z}$, there exists $t \in \mathbb{R}$, satisfying $\alpha n < |t| < \beta n$ such that in the cylinder V/A equipped with the Riemannian metric locally isometric to V, we have $d(nz, \exp(tX)) \leq C$ (where nz, and $\exp(tX)$ are viewed in the cylinder V/A).

Proof. It is enough to prove the lemma in the case $V = \mathbb{R}^d$ with the usual coordinates and metric, $\Gamma = \mathbb{Z}^d$ generated by the canonical basis vectors, $t \mapsto tX$ with dense image in $\mathbb{R}^d/\mathbb{Z}^d$, $\mathbb{Z}^d = \mathbb{Z} \oplus \mathbb{Z}^{d-1}$, and z the first vector of the canonical basis. As $t \mapsto tX$ has dense image in $\mathbb{R}^d/\mathbb{Z}^d$, the vector X is not orthogonal to z. We may assume that X and z are in the same connected component of the complement of the hyper-plane orthogonal to z. Let $0 \leq \theta < \pi/2$ be the angle between X and z. We may assume $n \in \mathbb{N}$. We choose $t = z^n/\cos\theta$. In the cylinder $\frac{\mathbb{R}^d}{\{0\}\times\mathbb{Z}^{d-1}}$, we have $d(nz, tX) \leq \sqrt{2}/2$. □

Proposition 2.21. *Let G be a connected Lie group and assume that the radical $\sqrt{G}$ of G is not linear. Then the fundamental group of G is distorted in the universal cover of G. Also there exists a distorted central $\mathbb{Z}$-extension E of G with E connected.*

Proof. For this proof let us set $R = \sqrt{G}$. Since R is non-linear, there exists by Lemma 2.12 a compact subgroup $K < R$ such that $[R, R] \cap K \neq \{e\}$. Therefore, according to Lemma 2.8, there is a non-trivial one-parameter subgroup $\phi : \mathbb{R} \to [R, R] \cap T_N$, where T_N is the maximal compact subgroup of the nilradical N of G, which is also the nilradical of R. The closure of $\phi(\mathbb{R})$ in T_N is a torus $T \subseteq T_N$. Notice that T_N is central in G because T_N is a normal subgroup of G with discrete automorphism group. According to Lemma 2.5, the sequence of inclusions of closed connected normal subgroups $T \subseteq R \subseteq G$ induces the commutative diagram,

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma \cap \tilde{R} & \longrightarrow & \tilde{R} & \longrightarrow & R & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma \cap \tilde{T} & \longrightarrow & \tilde{T} & \longrightarrow & T & \longrightarrow & 1,
\end{array}$$

where $\tilde{G}$, $\tilde{R}$, and $\tilde{T}$, are resp. the universal covers of G, R, and T, where Γ, $\Gamma \cap \tilde{R}$, and $\Gamma \cap \tilde{T}$, are central discrete subgroups, resp. isomorphic to the fundamental groups of G, R, and T, and where the vertical arrows are injections of closed normal subgroups. Let $\tilde{\phi}(t) = \exp(tX)$ be the one-parameter subgroup of $\tilde{T}$ which covers ϕ. Let $\mathfrak{r}$ denote the Lie algebra of $\tilde{R}$. The vector $X \neq 0$ belongs to $[\mathfrak{r}, \mathfrak{r}]$ and is central in the Lie algebra of $\tilde{G}$, hence we can apply Proposition 2.18, with $S = \tilde{R}$, $N = [S, S]$ and Z the one-parameter subgroup $\tilde{\phi}$ in $\tilde{T}$, to deduce that the one-parameter subgroup $\tilde{\phi}$ is distorted in $\tilde{R}$. As the image of $\tilde{\phi}$ in $\tilde{T} \cong \mathbb{R}^{dim(T)}$ is a line and as $\Gamma \cap \tilde{T} \cong \mathbb{Z}^{dim(T)}$ is cocompact in $\tilde{T}$, we deduce that $\Gamma \cap \tilde{T}$ is distorted in $\tilde{R}$, hence in $\tilde{G}$ as well. As $\Gamma \cap \tilde{T}$ is undistorted in Γ, this shows that Γ is distorted in $\tilde{G}$.

To prove the existence of a distorted topological $\mathbb{Z}$-extension of G, let us choose a non-zero element z_0 in the free abelian group $\Gamma \cap \tilde{T}$. It is possible to choose a direct summand $\mathbb{Z}$ in the finitely generated abelian group Γ, such that a generator z of the direct summand $\mathbb{Z}$ shares a non-zero power with z_0: there exist $m, n \in \mathbb{N}$, such that $z_0^m = z^n$. This implies that the infinite cyclic groups (z_0) and (z) lie at bounded distance from each other in $\tilde{G}$ (with respect to any left-invariant Riemannian metric on $\tilde{G}$). Let B be a complement of $(z) = \mathbb{Z}$ in Γ and let $E = \tilde{G}/B$. Let $p : \tilde{G} \to E$ denote the canonical projection. The connected Lie group E is a topological $\mathbb{Z}$-extension of G, with kernel generated by the image $p(z) \in E$. The proof will be finished if we prove that the kernel $(p(z)) \cong \mathbb{Z}$ is distorted in E. As $\tilde{T}$ is the universal cover of the closure T of ϕ, and as $\tilde{\phi}$ is distorted in $\tilde{G}$, we deduce from Lemma 2.16 that the embedding of $p(\tilde{\phi})$ in E is distorted in E, and from Lemma 2.20 (applied with $(p(z_0)) = \mathbb{Z}$) that $(p(z)) = \mathbb{Z}$ itself is distorted in E. □

Remark 2.22. *The above proof shows, under the hypothesis of Proposition 2.21, that the distortion function of the fundamental group in the universal cover of G grows at least linearly. Similarly, the kernel of the distorted topological $\mathbb{Z}$-extension E of G is at least linearly distorted.*

Remark 2.23. *In the example of Remark 2.11, the fundamental group $\pi_1(G)$ is distorted in the universal cover $\tilde{G}$ of G, but each infinite cyclic subgroup of $\pi_1(G)$ is undistorted in $\tilde{G}$. This last condition implies that the image of the one-parameter subgroup ϕ (in the proof of Proposition 2.21) is not closed.*

3. On Borel cohomology

3.1. **The relationship between the various cohomology groups.** In this subsection we recall the relationship we need between various cohomology groups. Recall that a map $f : X \to Y$ of topological spaces, is *Borel* if it is measurable with respect to the σ-algebras generated by the open subsets of X resp. Y. For G any topological group and A a metric abelian group, we write $C^n_B(G, A)$, resp. $C^n_{Bb}(G, A)$ for the group of Borel maps $G^n \to A$, resp. bounded Borel maps. With the usual differentials, this defines a cochain complexes $C^*_{B(b)}(G, A)$, whose respective cohomologies,

$$H^*_B(G, A) \quad \text{and} \quad H^*_{Bb}(G, A)$$

are the *Borel*, resp. *Borel bounded* cohomology of G with coefficients in A; if A is finitely generated abelian, we always assume that the metric on A corresponds to the word metric coming from a finite symmetric generating set. In a similar way, the groups H^*_c, H^*_{cb}, H^*_b are defined using cochains which are continuous, continuous bounded, or just bounded. We refer the reader to [29–31], [20], and [28] for the definition and functorial properties of these cohomology theories.

All groups will usually be supposed to be separable and locally compact, with topology given by a complete metric (occasionally we will also consider non-separable groups like $\mathbb{R}^\delta$).

Our main object of study is the forgetful map $H^*_{Bb}(G, \mathbb{Z}) \to H^*_B(G, \mathbb{Z})$ for the case of a virtually connected Lie group. Note that the target group $H^*_B(G, \mathbb{Z})$ is naturally isomorphic to the singular cohomology $H^*(BG, \mathbb{Z})$ of the classifying space BG of G (cf. [36, Theorem 4, p.93] for the proof and [1, Section 7] for some useful comments). There are also canonical isomorphisms $H^*_c(G, \mathbb{R}) \cong H^*_B(G, \mathbb{R})$ (see [36, Theorem 3, p. 91] and [30, Section 7. (2), p.32]), and $H^*_{cb}(G, \mathbb{R}) \cong H^*_{Bb}(G, \mathbb{R})$ (see [10, Section 2.3, (2i), p. 15] which refers to [2, Section 4]). We write G^δ for G considered as a discrete group. We have a canonical isomorphism $H^*(G^\delta, \mathbb{R}) \cong H^*(BG^\delta, \mathbb{R})$.

The relationship between the various cohomology groups can be expressed by the following natural commutative diagram:

$$
\begin{array}{ccccc}
H^*_{Bb}(G,\mathbb{Z}) & \longrightarrow & H^*_B(G,\mathbb{Z}) \cong H^*(BG,\mathbb{Z}) & \longrightarrow & H^*(BG,\mathbb{R}) \\
\Big\downarrow & & & & \Big\downarrow \phi \\
& & H^*_{cb}(G,\mathbb{R}) \cong H^*_{Bb}(G,\mathbb{R}) & \longrightarrow & H^*_c(G,\mathbb{R}) \cong H^*_B(G,\mathbb{R}) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
H^*_b(G^\delta,\mathbb{Z}) & \longrightarrow & H^*_b(G^\delta,\mathbb{R}) & \longrightarrow & H^*(G^\delta,\mathbb{R}).
\end{array}
$$

The map ϕ from the upper right corner is defined as the composition:

$$\phi : H^*(BG,\mathbb{R}) \cong H^*(BG,\mathbb{Z}) \otimes \mathbb{R} \cong H^*_B(G,\mathbb{Z}) \otimes \mathbb{R} \to H^*_B(G,\mathbb{R}). \tag{1}$$

3.2. **On Borel cohomology.** In this subsection, we recall and prove several general properties in bounded cohomology (Lemmata 3.1, 3.2, 3.5, 3.9, Corollaries 3.4, 3.6, Proposition 3.7). We also give examples of unbounded characteristic classes of any even degree (Example 3.13).

Lemma 3.1. *Let G be a topological group and $x \in H^d_B(G,\mathbb{Z})$ be such that for some $n > 0$, nx is bounded. Then x is bounded as well.*

Proof. The short exact coefficients sequence, $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, yields long exact cohomology sequences, for Borel bounded cohomology as well as for Borel cohomology:

$$
\begin{array}{ccccccc}
H^{d-1}_{Bb}(G,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^d_{Bb}(G,\mathbb{Z}) & \xrightarrow{n} & H^d_{Bb}(G,\mathbb{Z}) & \longrightarrow & H^d_{Bb}(G,\mathbb{Z}/n\mathbb{Z}) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
H^{d-1}_{B}(G,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^d_{B}(G,\mathbb{Z}) & \xrightarrow{n} & H^d_{B}(G,\mathbb{Z}) & \longrightarrow & H^d_{B}(G,\mathbb{Z}/nZ).
\end{array}
$$

Using that the vertical maps are isomorphisms for $\mathbb{Z}/n\mathbb{Z}$-coefficients, the result follows from a simple diagram chase. □

Lemma 3.2. *Let G be a topological group. Let x^δ be a characteristic class of G in the image of the composition $H^d(BG,\mathbb{Z}) \to H^d(BG,\mathbb{R}) \to H^d(BG^\delta,\mathbb{R})$. If x^δ is bounded then it admits a representative cocycle whose set of values on the singular simplices of BG^δ is a finite subset of $\mathbb{Z}$.*

Proof. The exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$ admits a bounded section. Let us focus on the following two horizontal exact sequences with commutative squares, it induces.

$$
\begin{array}{ccccc}
H^d_b(BG^\delta,\mathbb{Z}) & \longrightarrow & H^d_b(BG^\delta,\mathbb{R}) & \longrightarrow & H^d_b(BG^\delta,S^1) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow = \\
H^d(BG^\delta,\mathbb{Z}) & \longrightarrow & H^d(BG^\delta,\mathbb{R}) & \longrightarrow & H^d(BG^\delta,S^1)\,.
\end{array}
$$

By hypothesis, x^δ has a representative cocycle with integer values, hence it goes to zero in $H^d(BG^\delta,S^1)$. By hypothesis, it is the image of an element $w \in H^d_b(BG^\delta,\mathbb{R})$. A simple diagram chase shows that w is in the image of $H^d_b(BG^\delta,\mathbb{Z}) \to H^d_b(BG^\delta,\mathbb{R})$. □

Lemma 3.3. *Let $p : H \to G$ be a covering map of connected Lie groups. Then the induced map $H^*(BG, \mathbb{R}) \to H^*(BH, \mathbb{R})$ is surjective.*

Proof. As p is a $\pi_1(G)$-Galois covering and as $\pi_1(G)$ is finitely generated abelian, we can factor p in a sequence of connected covering spaces $H = X_0 \to X_1 \to \cdots X_n = G$ with $X_i \to X_{i+1}$ a cyclic covering. If $X_i \to X_{i+1}$ is a finite covering, the induced map $H^*(BX_{i+1}, \mathbb{R}) \to H^*(BX_i, \mathbb{R})$ is an isomorphism, because the fiber of $BX_i \to BX_{i+1}$ is $\mathbb{R}$-acyclic. In case $X_i \to X_{i+1}$ is an infinite cyclic covering, there is a circle fibration $S^1 \to BX_i \to BX_{i+1}$ with associated Gysin sequence with $\mathbb{R}$-coefficients

$$H^d(BX_{i+1}) \xrightarrow{\cup e} H^{d+2}(BX_{i+1}) \longrightarrow H^{d+2}(BX_i) \xrightarrow{\theta(d)} H^{d+1}(BX_{i+1})$$

in which $\theta(d)$ is the zero map, because the real cohomology of a connected Lie group is concentrated in even dimensions. It follows that $H^*(BX_{i+1}, \mathbb{R}) \to H^*(BX_i, \mathbb{R})$ is surjective and therefore the composite map $H^*(BG, \mathbb{R}) \to H^*(BH, \mathbb{R})$ is surjective too. □

This yields the following useful corollary.

Corollary 3.4. *Let $p : H \to G$ be a covering map of connected Lie groups. If all Borel cohomology classes in degree d with $\mathbb{Z}$ coefficients for G are bounded then they are all bounded for H as well.*

Proof. Using the natural isomorphism

$$H^*_B(L, \mathbb{Z}) \cong H^*(BL, \mathbb{Z})$$

for L a connected Lie group, we have a natural commutative diagram

$$\begin{array}{ccc} H^d_{Bb}(G, \mathbb{Z}) & \xrightarrow{\quad p^*_{Bb} \quad} & H^d_{Bb}(H, \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^d_B(G, \mathbb{Z}) = H^d(BG, \mathbb{Z}) & \xrightarrow{p^*_B} & H^d_B(H, \mathbb{Z}) = H^d(BH, \mathbb{Z}). \end{array}$$

In view of Lemma 3.3 we know that p^*_B maps onto a subgroup of finite index. Therefore, if all Borel cohomology classes in degree d for G are bounded, then for every $x \in H^d_B(H, \mathbb{Z})$ there is an $m > 0$ so that mx is bounded. But this implies by Lemma 3.1 that already x is bounded. □

Lemma 3.5. *Let G be a topological group. Suppose that the image of $x \in H^*_B(G, \mathbb{Z})$ in $H^*_B(G, \mathbb{R})$ is bounded, (i.e., lies in the image of $H^*_{Bb}(G, \mathbb{R}) \to H^*_B(G, \mathbb{R})$). Then x is bounded as well.*

Proof. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$ of topological groups admits a Borel bounded section $S^1 \to \mathbb{R}$ and gives therefore rise to long exact cohomology sequences and a commutative diagram

$$\begin{array}{ccccccc} \cdots H^{d-1}_{Bb}(G, S^1) & \longrightarrow & H^d_{Bb}(G, \mathbb{Z}) & \longrightarrow & H^d_{Bb}(G, \mathbb{R}) & \longrightarrow & H^d_{Bb}(G, S^1) \cdots \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ \cdots H^{d-1}_{B}(G, S^1) & \longrightarrow & H^d_{B}(G, \mathbb{Z}) & \longrightarrow & H^d_{B}(G, \mathbb{R}) & \longrightarrow & H^d_{B}(G, S^1) \cdots . \end{array}$$

A simple diagram chase completes the proof. □

Corollary 3.6. *If G is a compact Lie group, then all elements of $H^*_B(G,\mathbb{Z})$ are bounded.*

Proof. Let $x \in H^*_B(G,\mathbb{Z})$. Then its image $y \in H^*_B(G,\mathbb{R})$ is bounded, because by van Est's theorem, the continuous cohomology of a compact Lie group vanishes in positive dimensions, and $H^*_B(G,\mathbb{R}) = H^*_c(G,\mathbb{R})$ by Wigner [36, Theorem 3, p. 91]. □

Proposition 3.7. *Let G be a virtually connected Lie group and let G^0 be the connected component of the identity. If $x \in H^*_B(G,\mathbb{Z})$ restricts to a bounded class in $H^*_B(G^0,\mathbb{Z})$ then x itself is bounded. If $y \in H^*(BG^\delta,\mathbb{R})$ restricts to a class in $H^*(B(G^0)^\delta,\mathbb{R})$ which has a representing cocycle which takes only finitely many values then y has such a representative too.*

Proof. Assume $x \in H^*_B(G,\mathbb{Z})$ restricts to a bounded class in $H^*_B(G^0,\mathbb{Z})$. It suffices by Lemma 3.5 to show that the image $u = \gamma(x) \in H^*_B(G,\mathbb{R})$ of x is bounded. We have a natural commutative diagram with horizontal arrows β and δ induced by restriction:

$$\begin{array}{ccccc} H^d_{Bb}(G,\mathbb{Z}) & \xrightarrow{\alpha} & H^d_{Bb}(G,\mathbb{R}) & \xrightarrow[\cong]{\beta} & H^d_{Bb}(G^0,\mathbb{R})^{G/G^0} \\ \downarrow & & \downarrow \phi & & \downarrow \psi|\mathrm{inv} \\ x \in H^d_B(G,\mathbb{Z}) & \xrightarrow{\gamma} & H^d_B(G,\mathbb{R}) & \xrightarrow[\cong]{\delta} & H^d_B(G^0,\mathbb{R})^{G/G_0} . \end{array}$$

That the horizontal arrows β and δ are isomorphisms follows from the Lyndon-Hochschild-Serre spectral sequences for the short exact sequences $G^0 \to G \to G/G^0$, using that the Borel bounded cohomology and the Borel cohomology with $\mathbb{R}$ coefficients vanishes for the finite group G/G^0 in positive dimensions (for the Lyndon-Hochschild-Serre spectral sequence in continuous, resp. continuous bounded, cohomology see [2] resp. [28, Chapter IV, Section 12]; as we have mentioned at the beginning of Section 3, there are natural isomorphisms $H^*_B(G,\mathbb{R}) \cong H^*_c(G,\mathbb{R})$, resp. $H^*_{Bb}(G,\mathbb{R}) \cong H^*_{cb}(G,\mathbb{R})$). From our assumption it follows that $\delta(u) = v$ is bounded, say $v = \psi(w)$ for some $w \in H^d_{Bb}(G^0,\mathbb{R})$. By averaging with respect to the G/G^0-action we can form $\overline{w} = \frac{1}{|G:G^0|}\sum gw$ where the sum is taken over a set of coset representatives of G^0 in G. Then $\overline{w} \in H^d_{Bb}(G^0,\mathbb{R})^{G/G^0}$ and $\psi(\overline{w}) = v$. Thus $\beta^{-1}(\overline{w})$ is a bounded representative for u and we are done with the first case.

The case of $H^*(BG^\delta,\mathbb{R})$ can be dealt with in a similar way, using the fact that the restriction map induces an isomorphism $H^*(BG^\delta,\mathbb{R}) \to H^*(B(G^0)^\delta,\mathbb{R})^{G/G^0}$. □

Example 3.8. *The following is an example of a virtually connected Lie group G with all cohomology classes in $H^2_B(G,\mathbb{Z})$ bounded but having an unbounded class in $H^2_B(G^0,\mathbb{Z})$. Take $\mathbf{H}$ to be the three dimensional real Heisenberg group. It admits an involution $T : \mathbf{H} \to \mathbf{H}$ which induces multiplication by -1 on the center of $\mathbf{H}$. In matrix notation,*

$$T: \quad \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -a & -c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Then T preserves the infinite cyclic central subgroup $\mathbb{Z}$ generated by any chosen non-zero central element. Thus T induces an involution on $\mathbf{H}/\mathbb{Z}$, which induces *multiplication by* -1 *on* $H^2(B(\mathbf{H}/\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$. *It follows that the semi-direct product* $G := (\mathbf{H}/\mathbb{Z}) \rtimes_T \mathbb{Z}/2\mathbb{Z}$ *has* $H^2_B(G,\mathbb{Z})$ *finite, hence bounded according to Lemma 3.1 (it is a finite group isomorphic to* $H^2(BO(2),\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, *because* $H^2_B(G,\mathbb{Z}) \cong H^2(BG,\mathbb{Z})$ *and* G *has* $O(2)$ *as a maximal compact subgroup). Now let* $x \in H^2_B(G^0,\mathbb{Z}) \cong H^2(BG^0,\mathbb{Z}) \cong \mathbb{Z}$ *be a generator. We claim that* x *is not bounded. If* x *were bounded, all of* $H^*_B(G^0,\mathbb{Z})$ *were bounded, because* $H^*_B(G^0,\mathbb{Z})$ *is generated by an algebra by* x. *But this contradicts Theorem 1.1, because the closed subgroup* $[\sqrt{G^0},\sqrt{G^0}] \cong \mathbb{R}/\mathbb{Z}$ *is not simply-connected and thus* $\sqrt{G^0} = G^0$ *is not linear.*

Lemma 3.9. *Let* A *be a finitely generated abelian group. If* $f : G \to Q$ *is a homomorphism of Lie groups and a homotopy equivalence of the underlying topological spaces, then the induced map* $f^* : H^*_B(Q,A) \to H^*_B(G,A)$ *is an isomorphism. In particular, if all classes of* $H^*_B(Q,A)$ *are bounded, then the same is true for* $H^*_B(G,A)$.

Proof. The map f induces a homotopy equivalence of classifying spaces $Bf : BG \to BQ$. Indeed, it induces an isomorphism at the level of homotopy groups and the spaces in question have the homotopy type of CW-complexes. On the other hand, $H^*_B(G,A)$ is naturally isomorphic to $H^*(BG,A)$ (cf. [36]). This proves that f^* is an isomorphism. If c is a bounded representative of $[c] \in H^*_B(Q,A)$, then f^*c is a bounded representative of $f^*[c]$. This concludes the proof. □

For the convenience of the reader we give the proof of the following two results from Bucher-Karlsson's thesis.

Proposition 3.10. (Bucher-Karlsson [8, p. 60]) *Let* G *be a locally compact group with a cocompact lattice* Γ. *Let* $res : H^*_c(G,\mathbb{R}) \to H^*(\Gamma,\mathbb{R})$ *be the restriction map and let* $x \in H^*_c(G,\mathbb{R})$. *If* $res(x)$ *is bounded then* x *admits a continuous, bounded cocycle representative.*

Proof. There is a commutative diagram:

$$\begin{array}{ccc} H^*_{cb}(G,\mathbb{R}) & \xrightarrow{F} & H^*_c(G,\mathbb{R}) \\ \uparrow{\scriptstyle tr} & & \uparrow{\scriptstyle tr} \\ H^*_b(\Gamma,\mathbb{R}) & \xrightarrow{F} & H^*(\Gamma,\mathbb{R}), \end{array}$$

where F denotes forgetful maps from bounded cohomologies and tr denotes the transfer maps (cf. [28, p. 107]). The equality

$$tr \circ res = id,$$

holds on $H^*_c(G,\mathbb{R})$ (as well as on $H^*_{cb}(G,\mathbb{R})$). By hypothesis, there exists $y \in H^*_b(\Gamma,\mathbb{R})$ such that $F(y) = res(x)$. A continuous, bounded cocycle representative of the transfer of y exists by definition and

$$F(tr(y)) = tr(F(y)) = tr(res(x)) = x.$$

□

The map $\phi : H^*(BG,\mathbb{R}) \to H^*_c(G,\mathbb{R})$ in the following Proposition is the one defined in Subsection 3.1, Equation 1.

Proposition 3.11. (Bucher-Karlsson [8, p. 60].) *Let G be a Lie group of the form $G_a(\mathbb{R})^0$ for some semi-simple linear algebraic $\mathbb{R}$-group G_a. Every element in the image of $\phi : H^*(BG, \mathbb{R}) \to H^*_c(G, \mathbb{R})$ admits a continuous, bounded cocycle representative.*

Proof. Let $x \in H^*(BG, \mathbb{R})$. Its image $\phi(x)$ maps to the element $x^\delta \in H^*(BG^\delta, \mathbb{R}) \cong H^*(G^\delta, \mathbb{R})$. The proof of Gromov's result [17, Section 1.3, p. 23] by Bucher-Karlsson [9] works in a semi-algebraic setting, hence applies to $G = G_a(\mathbb{R})^0$. Therefore, the class x^δ is bounded. According to [3], there exists a cocompact lattice Γ in G. As the arrows of the following commutative diagram,

$$\begin{array}{ccc} H^*_c(G, \mathbb{R}) & & \\ \downarrow & \searrow^{res} & \\ H^*_b(G^\delta, \mathbb{R}) & \xrightarrow{res} & H^*(\Gamma, \mathbb{R}), \end{array}$$

preserve boundedness, the class $res(\phi(x)) \in H^*(\Gamma, \mathbb{R})$ is also bounded. Proposition 3.10 shows that the class $\phi(x)$ has a continuous, bounded representative. □

Translating this into the context of Borel cohomology and passing to Lie groups with finitely many connected components, we obtain the following.

Corollary 3.12. *Let G be a virtually connected Lie group with G^0 semi-simple (not necessarily linear). Then $H^*_{Bb}(G, \mathbb{Z}) \to H^*_B(G, \mathbb{Z})$ is surjective.*

Proof. We first reduce to the case of the connected component G^0 by Proposition 3.7. Dividing by the center, we reduce to the case of $G^0/Z(G^0)$ by using Corollary 3.4. Now $G^0/Z(G^0) := H$ is a connected linear semi-simple Lie group. According to [25] or [37, Proposition 3.1.6], H is as a Lie group isomorphic to $G_a(\mathbb{R})^0$ for some connected real linear algebraic group G_a. Let $x \in H^*_B(H, \mathbb{Z})$. According to Lemma 3.5, it is enough to show that the image $y \in H^*_B(H, \mathbb{R})$ of x is bounded. In view of the commutative diagram of Subsection 3.1, and using 3.11, the class y admits a continuous, bounded cocycle representative. □

The following family of examples of Lie groups with non-linear radical shows that one can have unbounded primary characteristic classes in any positive (even) degree. (The basic idea in the construction appears in [15] and also in [19, Remarques 7.5, b)].)

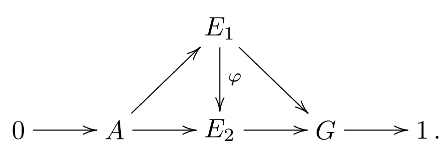
Example 3.13. *Let G_i be the i-fold cartesian product of Heisenberg quotients $\mathbf{H}/\mathbb{Z}$, where $\mathbf{H}$ is the 3-dimensional Heisenberg group and $\mathbb{Z} < \mathbf{H}$ a central subgroup. Note that the maximal compact subgroup of $\mathbf{H}/\mathbb{Z}$ is S^1 so that $H^*(BG_i, \mathbb{R})$ is a polynomial algebra generated by 2-dimensional classes x_j, $1 \leq j \leq i$, with the property that the map induced by the j-th injection $\mathbf{H}/\mathbb{Z} \to G_i$ maps x_j to a non-zero element in $H^2(B(\mathbf{H}/\mathbb{Z}), \mathbb{R})$. According to Goldman [15] there is a flat $\mathbf{H}/\mathbb{Z}$-bundle over the 2-torus T^2 given by a homomorphism $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbf{H}/\mathbb{Z}$ such that $\phi^* : H^2(B(\mathbf{H}/\mathbb{Z}), \mathbb{R}) \to H^2(T^2, \mathbb{R})$ is non-zero. Taking i-fold products, we get a flat G_i-bundle of the $2i$-torus T^{2i}, with associated map $H^*(BG_i, \mathbb{R}) \to H^*(T^{2i}, \mathbb{R})$ mapping the product $y_d := x_1 \cdots x_d$ non-trivially into $H^{2d}(T^{2i}, \mathbb{R})$, where $1 \leq d \leq i$. It follows that the images $y_d^\delta \in H^{2d}(BG_i^\delta, \mathbb{R})$ of y_d are all non-trivial. But they cannot have bounded cocycle representatives, because the bounded cohomology of the* *nilpotent group* G_i^δ *vanishes in positive dimensions. It also follows that for all* $i \geq 1$ *and all* $1 \leq d \leq i$,

$$H^{2d}_{Bb}(G_i, \mathbb{Z}) \to H^{2d}_B(G_i, \mathbb{Z})$$

is not surjective.

3.3. **Topological A-extensions.** In this subsection we define *topological A-extensions*. Moore's theorem (see Theorem 3.14) shows that they are closely related to 2-dimensional Borel cohomology. In Lemmata 3.15, 3.20, 3.21, we establish some properties of 2-cocycles associated to central extensions. In Proposition 3.23 we show that the 2-dimensional Borel cohomology with $\mathbb{Z}$-coefficients of a Lie group G is bounded if and only if the class defined by the universal cover of G is bounded. Proposition 3.25 shows that for a connected normal closed subgroup N of a connected Lie group G, each real class of degree 2 on the classifying space BN is the restriction of a class on BG.

A central extension of topological groups $0 \to A \to E \to G \to 1$ will be referred to as a *topological A-extension*. In this setting, we always assume that the groups are locally compact and second countable and that the monomorphism $A \to E$ has closed image. We write $\mathrm{Ext}_{top}(G, A)$ for the group of isomorphism classes of topological A-extensions of G, where two extensions $0 \to A \to E_1 \to G \to 1$ and $0 \to A \to E_2 \to G \to 1$ are called *isomorphic* if there is an isomorphism of topological groups $\varphi : E_1 \to E_2$ making the following diagram commute:



In case G is a connected Lie group and A is discrete, a topological A-extension is just a covering space $E \to G$ with covering transformation group A. Such an E is determined by a homomorphism $\phi : \pi_1(G) \to A$, with

$$E = A \times_\phi \tilde{G} = (A \times \tilde{G})/\pi_1(G)$$

with $\tilde{G}$ the universal cover of G, where $\pi_1(G)$ acts on $\tilde{G}$ via deck transformations and on A via ϕ. It follows that there is a natural isomorphism

$$\mathrm{Ext}_{top}(G, A) \cong \mathrm{Hom}(\pi_1(G), A)\,.$$

A topological A-extension $0 \to A \to E \to G \to 1$ always admits a Borel section σ (cf. [24]). We denote $c_\sigma : G \times G \to A$ the Borel 2-cocycle

$$c_\sigma(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}.$$

The following statement is a special case of [30, Theorem 10]:

Theorem 3.14. (Moore [30, Theorem 10].) *Let G and A be locally compact separable groups with A abelian. The map*

$$\begin{array}{rcl} \mathrm{Ext}_{top}(G, A) & \to & H^2_B(G, A) \\ \{0 \to A \to E \to G \to 1\} & \mapsto & [c_\sigma], \end{array}$$

is an isomorphism.

Lemma 3.15. *Let G and A be locally compact separable groups with A abelian. Let $c : G \times G \to A$ be a Borel 2-cocycle. There is a topological A-extension*

$$0 \to A \to E \to G \to 1$$

and a Borel section $\sigma : G \to E$ such that $c = c_\sigma$.

Proof. Let E a topological A-extension which represents $[c]$ via Moore's Theorem 3.14 and choose a Borel section τ of E then $[c] = [c_\tau]$. There exists a Borel map $b : G \to A$, such that $c = c_\tau + db$. Recall that the coboundary operator on a 1-cochain b, with A a the trivial G-module, is $db(g, g') = b(g') - b(gg') + b(g)$. The Borel cocycle c_σ, associated to the Borel section $\sigma : G \to E$, defined as $\sigma(g) = \tau(g)b(g)$, is equal to c. □

Lemma 3.16. *Let $0 \to A \to E \to G \to 1$, be an exact sequence of second countable locally compact groups, with A discrete in E. Let σ be a Borel section of the projection $p : E \to G$. Let μ_E be a Haar measure on E. There exists a Borel subset X of E with the following properties:*

1. *X is relatively compact,*
2. *$\mu_E(X) > 0$,*
3. *$X = X^{-1}$,*
4. *$\sigma(p(X))$ is relatively compact.*

Proof. Let U be an open symmetric relatively compact subset of E, small enough such that the covering projection p restricted to U, is a homeomorphism onto its image $V = p(U)$. Let μ_E, resp. μ_G, be the Haar measure on E, resp. on G, such that $\mu_E(U) = 1$, resp. such that $\mu_G(V) = 1$. The uniqueness of the Haar measure, together with the fact that the homomorphism p is a local homeomorphism, imply that for any Borel subset A of U, we have $\mu_E(A) = \mu_G(p(A))$ (see [5, Intégration, chapitre 7, paragraphe 2, numéro 7, proposition 10, p. 60]). Let $E = \cup_{n\in\mathbb{N}} K_n$, be an exhaustion of E by compact subsets, such that $K_n \subseteq K_{n+1}$. Let $B_n = \{v \in V : \sigma(v) \in K_n\} = V \cap \sigma^{-1}(K_n)$. It is a Borel subset of G because σ is a Borel map. Notice that

$$\lim_{n\to\infty} \mu_G(V \setminus B_n) = 0.$$

We conclude from this, and from the fact that the modular function of G is bounded on V, that there exists $N \in \mathbb{N}$, such that $\mu_G(V \setminus B_N) < 1/2$, and such that $\mu_G((V \setminus B_N)^{-1}) < 1/2$. Let

$$Y = V \setminus ((V \setminus B_N) \cup (V \setminus B_N)^{-1}).$$

The set Y has the following properties:

1. Y is relatively compact,
2. $\mu_G(Y) > 0$,
3. $Y = Y^{-1}$,
4. $\sigma(Y) \subseteq K_N$.

Let $X = p^{-1}(Y) \cap V$. The set X has the required properties. □

Remark 3.17. *A projection of separable locally compact groups always admits a locally bounded Borel section* [24, Lemma 2]*. The point in the above lemma is that the given Borel section σ is not assumed to be locally bounded.*

The following proposition has first been proved by Gersten [14], in the setting of finitely generated groups.

Proposition 3.18. *Let $0 \to A \to E \to G \to 1$, be an exact sequence of Lie groups, with E connected, A finitely generated discrete (hence central) in E. Let σ be a Borel section of the projection $p : E \to G$, with $\sigma(e) = e$. Let $c : G \times G \to A$, $c(g, g') = \sigma(g)\sigma(g')\sigma(gg')^{-1}$ be the corresponding cocycle. If c is bounded then A is undistorted in E.*

Proof. Let $x \in E$. There exist a unique element $a = a(x) \in A$ and a unique element $g = g(x) \in G$ such that $x = a\sigma(g)$. Let $X \subseteq E$ be as in Lemma 3.16. The set $X(\sigma(p(X)))^{-1}$, is relatively compact in E. This implies, together with the hypothesis that A is discrete in E, that there exists a finite symmetric generating set S of A with the property that if $x \in X$, then $a(x) \in S$.

As $\mu_E(X) > 0$, the set $XX^{-1} = XX$ contains a neighborhood of the identity of E (see [21, Chapter XII, Section 61, Exercise 3, p. 268]). As E is connected, the set X generates E.

Let $a \in A$ of length relative to X equal to n. Hence, there exist $x_1, \dots, x_n \in X$ such that

$$\begin{aligned} a &= x_1 \cdots x_n = a_1\sigma(g_1) \cdots a_n\sigma(g_n) \\ &= a_1 \cdots a_n c(g_1, g_2) \cdots c(g_1 \cdots g_{n-1}, g_n)\sigma(g_1 \cdots g_n). \end{aligned}$$

Applying p to the above equality, we get

$$e = p(a_1 \cdots a_n c(g_1, g_2) \cdots c(g_1 \cdots g_{n-1}, g_n)\sigma(g_1 \cdots g_n)) = g_1 \cdots g_n.$$

Since $\sigma(e) = e$, we see that the length of a relative to S is bounded by $n + (n - 1)\sup\{L_S(c(g, g'))|L_{p(X)}(g) < n, L_{p(X)}(g') \le 1\}$. □

Remark 3.19. *The proof of Proposition 3.18 shows in fact that the distortion of A in E is bounded by the growth of c.*

Lemma 3.20. *Let $p : G \to Q$ and $\pi : \tilde{Q} \to Q$ be surjective homomorphisms between topological groups.*

(1) *The total space of the pull-back $p^*(\tilde{Q}) \to G$ of $\tilde{Q} \to Q$ is a closed subgroup of $G \times \tilde{Q}$ and there is a short exact sequence of topological groups*
$$1 \to \ker(p) \to p^*(\tilde{Q}) \to \tilde{Q} \to 1.$$
(2) *Assume that both $\ker \pi$ in $\tilde{Q}$ and ker(p) in G are central. If the extension*
$$0 \to \ker(\pi) \to \tilde{Q} \to Q \to 1$$
can be defined by a Borel bounded cocycle, then the same is true for the extension
$$0 \to \ker(\pi) \to p^*(\tilde{Q}) \to G \to 1.$$

Proof. To prove (1), recall that by definition,

$$p^*(\tilde{Q}) = \{(g, \tilde{q}) \in G \times \tilde{Q} \text{ such that } p(g) = \pi(\tilde{q})\}.$$

We embed $ker(p)$ in $p^*(\tilde{Q})$ by composing the inclusions $ker(p) \subset G \subset G \times \{e\}$. The map $\tilde{p} : p^*(\tilde{Q}) \to \tilde{Q}$, defined as $(g, \tilde{q}) \mapsto \tilde{q}$, gives the wanted exact sequence. To prove (2), we apply Lemma 3.15 in order to obtain a Borel section $\sigma : Q \to \tilde{Q}$ of π such that the associated cocycle c_σ is bounded. Then $\sigma^* : G \to p^*(\tilde{Q})$ defined as $g \mapsto (g, \sigma p(g))$, is a Borel section of the projection $p^*(\tilde{Q}) \to G$ and its associated cocycle satisfies

$$c_{\sigma^*}(g, g') = (e, c_\sigma(p(g), p(g'))).$$

□

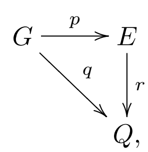
Lemma 3.21. *Let $p : X \to Y$ and $q : Y \to Z$ be two surjective homomorphisms of groups with central kernels. Let σ be a section of p, and τ one for q. Assume that the associated cocycles c_σ and c_τ have finite range. If the image under σ of the kernel of q is central in X then the cocycle $c_{\sigma\tau}$ associated to the section $\sigma\tau$ of qp has finite range.*

Proof. Since $\sigma(\ker(q))$ is central, the kernel of qp is central so that $qp : X \to Z$ is a central $\ker(qp)$ extension. One verifies that

$$c_{\sigma\tau}(x, y) = c_\sigma(\tau(x), \tau(y)) \cdot \sigma(c_\tau(x, y)) \cdot [c_\sigma(c_\tau(x, y), \tau(xy))]^{-1}.$$

Thus $c_{\sigma\tau}$ is a product of three functions each of which takes only finitely many values. It follows that $c_{\sigma\tau}$ has finite range. □

Remark 3.22. *Consider the following commutative diagram of groups*



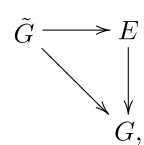
with q and r surjective. If σ is a section of q then $p\sigma$ is a section of r and

$$\begin{aligned} c_{p\sigma}(x, y) &= p\sigma(x)p\sigma(y)\,(p\sigma(xy))^{-1} \\ &= p\left(\sigma(x)\sigma(y)\,(\sigma(xy))^{-1}\right) = (p \circ c_\sigma)(x, y). \end{aligned}$$

Proposition 3.23. *Let G be connected Lie group. The following conditions are equivalent.*

1. *The class in $H^2_B(G, \pi_1(G))$, corresponding (in the sense of Theorem 3.14) to the universal cover of G, is bounded.*
2. *All classes in $H^2_B(G, \mathbb{Z})$ are bounded.*
3. *For every finitely generated abelian group A, all classes in $H^2_B(G, A)$ are bounded.*

Proof. We first prove that (1) implies (3). Let $x \in H^2_B(G, A)$ and let E be a central extension of G with central discrete kernel A corresponding to x. Let E^0 be the connected component of E. The restriction of the projection $E \to G$ to E^0 is a connected cover of G. Hence, we obtain a commutative diagram of covering groups,



where $\tilde{G}$ is the universal cover of G. Applying the hypothesis together with Remark 3.22, we deduce the existence of a Borel section $\tau : G \to E^0 \subseteq E$ of the projection $E \to G$ such that the corresponding cocycle $c_\tau : G \times G \to A$ is bounded. As $[c_\tau] = x$, we conclude that x is bounded.

The fact that (3) implies (2) is trivial and obviously (3) implies (1), since $\pi_1(G)$ is a finitely generated abelian group. It remains to see that (2) implies (3). Let $A = \mathbb{Z}^n \times F$ be a finitely generated abelian group, F a finite group, and let c :

$G \times G \to A$ be a 2-cocycle; it has components $c_i : G \times G \to \mathbb{Z}$, for $1 \le i \le n$ and a component c_F, taking values in F. The latter is obviously bounded. By assumption, the cocycles c_i are cohomologous to bounded cocycles d_i. Thus, the cocycle $G \times G \to A$ with components d_i and c_F is bounded and cohomologous to c, finishing the proof. □

Lemma 3.24. *If G and H are two virtually connected Lie groups such that the maps $H^2_{Bb}(G,\mathbb{Z}) \to H^2_B(G,\mathbb{Z})$ and $H^2_{Bb}(H,\mathbb{Z}) \to H^2_B(H,\mathbb{Z})$ are onto then so is the map $H^2_{Bb}(G \times H,\mathbb{Z}) \to H^2_B(G \times H,\mathbb{Z})$.*

Proof. Because $\pi_1(BG) = \pi_0(G)$ is finite, $H^1(BG,\mathbb{Z}) = 0$, and similarly $H^1(BH,\mathbb{Z}) = 0$. By the Künneth formula in singular cohomology, we conclude that the natural map $BG \sqcup BH \to BG \times BH = B(G \times H)$ induces an isomorphism ψ at the level of degree 2 cohomology with $\mathbb{Z}$ coefficients. On the other hand, the inclusion $i_G : G \to G \times H$ induces a surjection i_G^* in Borel bounded cohomology (because there is a retraction $G \times H \to G$), and similarly for the inclusion i_H. The following diagram commutes because for each component of the vertical maps we have a natural commutative diagram:

$$\begin{array}{ccc} H^2_{Bb}(G \times H,\mathbb{Z}) & \longrightarrow & H^2_B(G \times H,\mathbb{Z}) = H^2(B(G \times H),\mathbb{Z}) \\ {\scriptstyle \text{epi}}\downarrow{\scriptstyle (i_G^*, i_H^*)} & & {\scriptstyle \text{iso}}\downarrow{\scriptstyle \psi} \\ H^2_{Bb}(G,\mathbb{Z}) \oplus H^2_{Bb}(H,\mathbb{Z}) & \xrightarrow{\text{epi}} & H^2(BG,\mathbb{Z}) \oplus H^2(BH,\mathbb{Z}). \end{array}$$

This shows that the map $H^2_{Bb}(G \times H,\mathbb{Z}) \to H^2_B(G \times H,\mathbb{Z})$ is onto as well. □

Proposition 3.25. *Let G be a connected Lie group and let N be a closed connected normal subgroup of G. The natural map*

$$H^2(BG,\mathbb{R}) \to H^2(BN,\mathbb{R}),$$

induced by the inclusion $N \subset G$, is onto.

Proof. There is a commutative diagram:

$$\begin{array}{ccc} H^2(BG,\mathbb{Z}) & \longrightarrow & H^2(BN,\mathbb{Z}) \\ {\scriptstyle \cong}\downarrow & & {\scriptstyle \cong}\downarrow \\ Hom(\pi_1(G),\mathbb{Z}) & \longrightarrow & Hom(\pi_1(N),\mathbb{Z}). \end{array}$$

Tensoring with $\mathbb{R}$, we see that the proof will be finished if we show that

$$Hom(\pi_1(G),\mathbb{R}) \to Hom(\pi_1(N),\mathbb{R})$$

is onto. But this is the case according to Remark 2.4. □

4. Proof of the main results

4.1. Proof of some auxiliary results.

Lemma 4.1. *Let G be a connected Lie group an $\sqrt{G}$ its radical. The restriction map $H^*(BG,\mathbb{R}) \to H^*(B\sqrt{G},\mathbb{R})$ is surjective.*

Proof. According to Proposition 3.25, the map

$$H^2(BG,\mathbb{R}) \to H^2(B\sqrt{G},\mathbb{R})$$

is surjective. Because $\sqrt{G}$ is homotopy equivalent to a torus, the cohomology $H^*(B\sqrt{G},\mathbb{R})$ is a polynomial algebra on 2-dimensional classes. Therefore, since the restriction map $H^*(BG,\mathbb{R}) \to H^*(B\sqrt{G},\mathbb{R})$ is a map of algebras, it is surjective. □

Lemma 4.2. *Let G be a connected Lie group and $\sqrt{G}$ its radical. Then $H^*(BG,\mathbb{R})$ is generated as an algebra by $H^2(BG,\mathbb{R})$ together with the image of the inflation map $H^*(B(G/\sqrt{G}),\mathbb{R}) \to H^*(BG,\mathbb{R})$.*

Proof. Let I^* denote the image of $H^*(B(G/\sqrt{G}),\mathbb{R}) \to H^*(BG,\mathbb{R})$. By Lemma 4.1, the restriction map $H^*(BG,\mathbb{R}) \to H^*(B\sqrt{G},\mathbb{R})$ is surjective and therefore the fiber in the fibration $B\sqrt{G} \to BG \to BG/\sqrt{G}$ is totally non-homologous to zero relative to $\mathbb{R}$. We conclude from Borel's Theorem [4, Theorem 14.2]) that $H^*(BG,\mathbb{R})/\langle I^+\rangle$ maps isomorphically onto $H^*(B\sqrt{G},\mathbb{R})$, where $\langle I^+\rangle$ stands for the ideal generated by the classes $I^+ < I^*$ of positive degree. Using the well-known fact that for a connected Lie group L, $H^*(BL,\mathbb{R})$ is a polynomial algebra on even dimensional generators (see [4, Theorem 18.1]) we now prove our claim on the generation of $H^*(BG,\mathbb{R})$ by checking it for generators in even degrees. Thus, assume that $x \in H^{2d}(BG,\mathbb{R})$ is a generator with $d > 1$. Then x maps to an element $y \in H^{2d}(B\sqrt{G},\mathbb{R})$, which is a product of d 2-dimensional classes y_i. By choosing counter images x_i in $H^2(BG,\mathbb{R})$ for the y_i's, their product will be an element $z \in H^{2d}(BG,\mathbb{R})$ with $x - z \in \langle I^+\rangle$, say $x - z = \sum a_j b_j$ with $a_j \in I^+$ and $b_j \in H^{<2d}(BG,\mathbb{R})$. Thus $x = z + \sum a_j b_j$ lies in the subring generated by $H^2(BG,\mathbb{R})$ together with $\langle I^+\rangle$, proving the theorem. □

Lemma 4.3. *Let G be a connected Lie group. Assume that the radical $\sqrt{G}$ of G is linear. There is a closed contractible subgroup $V < \sqrt{G}$ which is normal in G such that $\sqrt{G}/V$ is a torus T and the covering space ξ induced from the universal cover $\tilde{L} \to G/\sqrt{G}$ via the canonical projection $G/V \to G/\sqrt{G}$ is of the form $\xi : T \times \tilde{L} \to G/V$.*

Proof. Let H be the covering space of G induced from the universal cover $\tilde{L}$ of $G/\sqrt{G}$. Then, according to (1) of Lemma 3.20 and Lemma 2.13, $H = \sqrt{G} \rtimes \tilde{L}$, with $\tilde{L}$ semi-simple. Let W be the linearizer of $\tilde{L}$; it is discrete and central in $\tilde{L}$. Also W is central in H because W lies in the kernel of the adjoint representation of H. So we can form the quotient

$$Q := \sqrt{G} \rtimes (\tilde{L}/W),$$

which is a linear Lie group by Theorem 2.1. According to Hochschild [22, XVIII.4, Theorem 4.3], the group Q contains a contractible normal closed solvable subgroup X with Q/X linear reductive. Since $X < \sqrt{G}$ we can think of X as a subgroup of $H = \sqrt{G} \rtimes \tilde{L}$, and this subgroup is normal, since W is central. Under the covering projection $H \to G$, this subgroup X maps isomorphically onto a closed normal subgroup $V < G$, which lies in the radical of G. Since Q/X is linear reductive, $\sqrt{G}/X$ is a central torus T (see [22, XVIII.4, Theorem 4.4]). Let ξ be the covering induced from the universal cover $\tilde{L} \to G/\sqrt{G}$ via the canonical projection $G/V \to G/\sqrt{G}$. According to (1) of Lemma 3.20, the total space E of ξ sits in the short exact sequence

$$0 \to T \to E \to \tilde{L} \to 1.$$

As $\tilde{L}$ is simply-connected, and as T is the radical of E, the sequence splits (again by Lemma 2.13). Since T is central, we deduce that E is a direct product $E = T \times \tilde{L}$. □

4.2. **Proof of Theorem 1.1.**

4.2.1. *(1)⇒(2) in Theorem 1.1.*

Proof. We first prove that all classes in $H^2_B(G,\mathbb{Z})$ are bounded. Since $\sqrt{G}$ is linear, Lemma 4.3 implies that G contains a normal contractible subgroup V, hence $G \to G/V$ is a homotopy equivalence, and thus $H^*_B(G/V,\mathbb{Z}) \to H^*_B(G,\mathbb{Z})$ an isomorphism by Lemma 3.9. We can therefore replace G by G/V. In the notation of Lemma 4.3, we have covering spaces $\mathbb{R}^n \times \tilde{L} \to T \times \tilde{L} \to G/V$. Note that $\mathbb{R}^n \times \tilde{L}$ is the universal cover of G/V. By Proposition 3.23, it suffices to show that the cocycle defining this universal cover can be chosen bounded. For this, we appeal to Lemma 3.21, with $X = \mathbb{R}^n \times \tilde{L}$, $Y = T \times \tilde{L}$ and $Z = G/V$. The map $p : X \to Y$ is the product of $\mathbb{R}^n \to T$ with $\mathrm{Id} : \tilde{L} \to \tilde{L}$ and we conclude from Lemma 3.24 and Corollary 3.6 that the Borel cocycle associated with p can be chosen to be bounded. For $q : Y \to Z$ we observe that q is obtained by pulling back $\tilde{L} \to G/\sqrt{G}$ over G/V. According to (2) of Lemma 3.20, it suffices to prove that the universal cover $\tilde{L} \to G/\sqrt{G}$ has a Borel section σ such that the associated cocycle c_σ is bounded. But this case has already been dealt with in Corollary 3.12 and it follows that $H^2_{Bb}(G,\mathbb{Z}) \to H^2_B(G,\mathbb{Z})$ is surjective. To pass to higher degrees, we will make use of Lemma 4.2 Since for every d, $H^d(BG,\mathbb{Z})$ is a finitely generated abelian group, we see that the $\mathbb{Z}$-subalgebra $A^* \subseteq H^*(BG,\mathbb{R})$ generated by $H^2(BG,\mathbb{Z})$ together with the image J^* of the inflation map $H^*(B(G/\sqrt{G}),\mathbb{Z}) \to H^*(BG,\mathbb{Z})$ satisfies, in view of Lemma 4.2, $A^* \otimes \mathbb{R} = H^*(BG,\mathbb{R})$. Thus $A^d \subseteq H^d(BG,\mathbb{Z})$ has finite index for every d. To show that $H^d_{Bb}(G,\mathbb{Z}) \to H^d_B(G,\mathbb{Z})$ is onto it suffices therefore by Lemma 3.1 to show that the elements of A^d all lie in the image of $H^d_{Bb}(G,\mathbb{Z}) \to H^d_B(G,\mathbb{Z})$. From the definition of A^* it suffices therefore to show that:

(a) the map $H^2_{Bb}(G,\mathbb{Z}) \to H^2_B(G,\mathbb{Z})$ is onto;
(b) the map $H^*_{Bb}(G/\sqrt{G},\mathbb{Z}) \to H^*_B(G/\sqrt{G},\mathbb{Z})$ is onto.

Part (a) we just proved and part (b) follows from Corollary 3.12, since $G/\sqrt{G}$ is semi-simple. □

4.2.2. *(2)⇒(3) in Theorem 1.1 .*

Proof. By Proposition 3.23 we conclude that the class defined by the universal cover is bounded. Let $A = \pi_1(G)$ be the fundamental group of G. We have a topological A-extension:

$$0 \to A \to \tilde{G} \to G \to 1,$$

where $\tilde{G}$ denotes the universal cover of G. Let τ be a Borel section of the projection $\tilde{G} \to G$. By hypothesis the class $[c_\tau] \in H^2_B(G,A)$ is bounded. According to Lemma 3.15, there exists a Borel section σ of $\tilde{G} \to G$ such that c_σ is bounded. We may modify σ at the identity so that $\sigma(e) = e$. It is easy to check that the new cocycle

c_σ associated to the modified section σ is still bounded. Proposition 3.18 implies that A is undistorted in $\tilde{G}$. □

4.2.3. *(3)⇒ (1) in Theorem 1.1.*

Proof. Suppose that $\sqrt{G}$ is not linear. Then according to Proposition 2.21, the fundamental group of G is distorted in $\tilde{G}$, contradicting assumption (3). □

The proof presented in 4.2.2 shows that if $H^2_B(G,\mathbb{Z})$ is bounded, then (3) in Theorem 1.1 holds. Using Proposition 3.23, we deduce that the equivalent conditions of Theorem 1.1 are also equivalent to the ones of the following Corollary.

Corollary 4.4. *Let G be a connected Lie group. The following are equivalent.*

1. *(1) Each Borel cohomology class of G with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.*
2. *(2) Each Borel cohomology class of G of degree two with $\mathbb{Z}$-coefficients can be represented by a Borel bounded cocycle.*
3. *(3) The class in $H^2_B(G,\pi_1(G))$ defined by the universal cover of G can be represented by a Borel bounded cocycle.*

5. Relationship with Gromov's Theorem

For the proof of Theorem 1.1. we made use of Gromov's Theorem on the boundedness of primary characteristic classes of flat G-bundles, where G is the Lie group of $\mathbb{R}$-points of a connected linear algebraic group over $\mathbb{R}$ (cf. [17, Section 1.3, p. 23]). On the other hand, our Theorem 1.1 implies the following generalization of Gromov's Theorem.

Theorem 5.1. *Let G be a virtually connected Lie group with linear radical. The image of the natural map*

$$H^*(BG,\mathbb{R}) \to H^*(BG^\delta,\mathbb{R})$$

consists of bounded classes.

Proof. Using Proposition 3.7 and because $\sqrt{G} = \sqrt{G^0}$ it suffices to consider the case of a connected group G. Thus, By Theorem 1.1, $H^*_{Bb}(G,\mathbb{Z}) \to H^*_B(G,\mathbb{Z})$ is surjective. From the commutative diagram displayed in Section 3.1 we then infer that the image of $H^*(BG,\mathbb{R}) \to H^*(BG^\delta,\mathbb{R})$ is contained in the image of $H^*_b(G^\delta,\mathbb{R}) \to H^*(G^\delta,\mathbb{R})$. □

Remark 5.2. *The converse of the previous result does not hold; that is, from the boundedness of primary real characteristic classes for flat G-bundles on cannot conclude that $\sqrt{G}$ is linear. For instance, if $G = S^1 \times \mathbf{H}/\mathbb{Z}$ is the group discussed in Remark 2.11 then, because G is solvable with $\pi_1([G,G]) = 0$, Goldman's result* [15] *implies that $\phi : H^*(BG,\mathbb{R}) \to H^*(BG^\delta,\mathbb{R})$ is trivial in positive dimensions and, therefore, the image of ϕ consist for trivial reasons of bounded classes. But $\pi_1\left(\overline{[G,G]}\right) \neq 0$, which implies that $\sqrt{G} = G$ is non-linear.*

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