ATIYAH'S L^2-INDEX THEOREM

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1. Introduction

The L^2-Index Theorem of Atiyah [1] expresses the index of an elliptic operator on a closed manifold M in terms of the G-equivariant index of some regular covering $\widetilde{M}$ of M, with G the group of covering transformations. Atiyah's proof is analytic in nature. Our proof is algebraic and involves an embedding of a given group into an acyclic one, together with naturality properties of the indices.

2. Review of the L^2-Index Theorem

The main reference for this section is Atiyah's paper [1]. All manifolds considered are smooth Riemannian, without boundary. Covering spaces of manifolds carry the induced smooth and Riemannian structure. Let M be a closed manifold and let E, F denote two complex (Hermitian) vector bundles over M. Consider an elliptic pseudo-differential operator

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

acting on the smooth sections of the vector bundles. One defines its space of solutions

$$S_D = \{s \in C^\infty(M, E) \mid Ds = 0\}.$$

The complex vector space S_D has finite dimension (see [13]), and so has S_{D^*} the space of solutions of the adjoint D^* of D where

$$D^* : C^\infty(M, F) \to C^\infty(M, E)$$

is the unique continuous linear map satisfying

$$\langle Ds, s' \rangle = \int_M \langle Ds(m), s'(m) \rangle_F \, dm = \langle s, D^* s' \rangle = \int_M \langle s(m), D^* s'(m) \rangle_E \, dm$$

for all $s \in C^\infty(M, E)$, $s' \in C^\infty(M, F)$. One now defines the *index* of D as follows:

$$\mathrm{Index}(D) = \dim_{\mathbb{C}}(S_D) - \dim_{\mathbb{C}}(S_{D^*}) \in \mathbb{Z}.$$

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An explicit formula for Index(D) is given by the famous Atiyah-Singer Theorem (cf. [2]). Consider a not necessarily connected, regular covering $\pi : \widetilde{M} \to M$ with countable covering transformation group G. The projection π can be used to define an elliptic operator

$$\widetilde{D} := \pi^*(D) : C_c^\infty(\widetilde{M}, \pi^*E) \to C_c^\infty(\widetilde{M}, \pi^*F).$$

Denote by $S_{\widetilde{D}}$ the closure of $\{s \in C_c^\infty(\widetilde{M}, \pi^*E) | \widetilde{D}s = 0\}$ in $L^2(\widetilde{M}, \pi^*E)$. Let $\widetilde{D}^*$ denote the adjoint of $\widetilde{D}$. The space $S_{\widetilde{D}}$ is not necessarily finite dimensional, but being a closed G-invariant subspace of the L^2-completion $L^2(\widetilde{M}, \pi^*E)$ of the space of smooth sections with compact supports $C_c^\infty(\widetilde{M}, \pi^*E)$, its von Neumann dimension is therefore defined as follows. Write

$$\mathcal{N}(G) = \{P : \ell^2(G) \to \ell^2(G) \text{ bounded and } G\text{-invariant}\}$$

for the group von Neumann algebra of G, where G acts on $\ell^2(G)$ via the right regular representation. Then $S_{\widetilde{D}}$ is a finitely generated Hilbert G-module and hence can be represented by an idempotent matrix $P = (p_{ij}) \in M_n(\mathcal{N}(G))$ (recall that a finitely generated Hilbert G-module is isometrically G-isomorphic to a Hilbert G-subspace of the Hilbert space $\ell^2(G)^n$ for some $n \geq 1$, see [9]). One then sets

$$\dim_G(S_{\widetilde{D}}) = \sum_{i=1}^{n} \langle p_{ii}(e), e\rangle = \kappa(P) \in \mathbb{R},$$

where by abuse of notation e denotes the element in $\ell^2(G)$ taking value 1 on the neutral element $e \in G$ and 0 elsewhere (see Eckmann's survey [9] on L^2-cohomology for more on von Neumann dimensions). The map $\kappa : M_n(\mathcal{N}(G)) \to \mathbb{C}$ is the Kaplansky trace. One defines the *L^2-index* of $\widetilde{D}$ by

$$\mathrm{Index}_G(\widetilde{D}) = \dim_G(S_{\widetilde{D}}) - \dim_G(S_{\widetilde{D}^*}).$$

We can now state Atiyah's L^2-Index Theorem.

Theorem 2.1 (Atiyah [1]). *For D an elliptic pseudo-differential operator on a closed Riemannian manifold M*

$$\mathrm{Index}(D) = \mathrm{Index}_G(\widetilde{D})$$

for any countable group G and any lift $\widetilde{D}$ of D to a regular G-cover $\widetilde{M}$ of M.

In particular, the L^2-index of $\widetilde{D}$ is always an integer, even though it is a priori given in terms of real numbers. The following serves as an illustration of the L^2-Index Theorem.

Example 2.2 (Atiyah's formula [1]). Let $\Omega^\bullet$ be the de Rham complex of complex valued differential forms on the closed connected manifold M and consider the de Rham differential $D = d + d^* : \Omega^{ev} \to \Omega^{odd}$. Let $\pi : \widetilde{M} \to M$ be the universal cover of M so that $G = \pi_1(M)$. Then

- $\mathrm{Index}(D) = \chi(M)$, the ordinary Euler characteristic of M.
- $\mathrm{Index}_G(\widetilde{D}) = \sum_j (-1)^j \beta^j(M)$, the L^2-Euler characteristic of M.

The $\beta^j(M)$'s denote the L^2-Betti numbers of M. Thus the L^2-Index Theorem translates into Atiyah's formula

$$\chi(M) = \sum_j (-1)^j \beta^j(M).$$

We recall that the L^2-Betti numbers $\beta^j(M)$ are in general not integers. For instance, if $\pi_1(M)$ is a finite group, one checks that

$$\beta^j(M) = \frac{1}{|\pi_1(M)|} b^j(\widetilde{M}),$$

where $b^j(\widetilde{M})$ stands for the ordinary j'th Betti number of the universal cover $\widetilde{M}$ of M. In particular, for $1 < |\pi_1(M)| < \infty$, $\beta^0(M) = 1/|\pi_1(M)|$ is not an integer and the L^2-Index Theorem reduces to the well-known fact that

$$\chi(M) = \frac{\chi(\widetilde{M})}{|\pi_1(M)|}.$$

It is a conjecture (Atiyah Conjecture) that for a general closed connected manifold M the L^2-Betti numbers $\beta^j(M)$ are always rational numbers, and even integers in case that $\pi_1(M)$ is torsion-free. For some interesting examples, which might lead to counterexamples, see Dicks and Schick [8].

3. Hilbert modules

Recall that for $H < G$ and X an H-space, the *induced* G-space is

$$G \times_H X = (G \times X)/H$$

where H acts on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$ and the left G-action on $G \times_H X$ is given by $g \cdot [k, x] = [gk, x]$ (where $[k, x]$ denotes the class of the pair $(k, x) \in G \times X$ in $G \times_H X$). For $A \subseteq \ell^2(H)^n$ a Hilbert H-module one defines $\mathrm{Ind}_H^G(A)$ the *induced* Hilbert G-module as follows:

$$\mathrm{Ind}_H^G(A) = \{f : G \to A, f(gh) = h^{-1} f(g), \sum_{\gamma \in G/H} \|f(\gamma)\|^2 < \infty\}.$$

On $\mathrm{Ind}_H^G(A)$ the action of G is given as follows:

$$(\gamma \cdot f)(\mu) = f(\gamma^{-1}\mu), \qquad \gamma, \mu \in G \text{ and } f \in \mathrm{Ind}_H^G(A).$$

For $\widetilde{M}$ an H-free, cocompact Riemannian manifold and $\widetilde{D}$ an H-equivariant pseudo-differential operator on $\widetilde{M}$, one can express the lift $\overline{D}$ of $\widetilde{D}$ to $\overline{M} = G \times_H \widetilde{M}$ as follows. Fix a set R of representatives for G/H and write $\pi : \overline{M} \to \widetilde{M}$ for the projection; a section $\overline{s} \in C_c^\infty(\overline{M}, \pi^* E)$ is a collection

$$\overline{s} = \{\widetilde{s}_r\}_{r \in R},$$

where $\widetilde{s}_r \in C_c^\infty(\widetilde{M}, E)$ is the zero section for all but finitely many r's, and $\overline{s}([g, \widetilde{m}]) = \widetilde{s}_r(h\widetilde{m})$, if $[r, h\widetilde{m}] = [g, \widetilde{m}] \in G \times_H \widetilde{M}$. Now the lift $\overline{D}$ of $\widetilde{D}$ to $\overline{M} = G \times_H \widetilde{M}$ satisfies

$$\overline{D}\overline{s} = \{\widetilde{D}\widetilde{s}_r\}_{r \in R}.$$

Lemma 3.1. *Let M be a closed Riemannian manifold, D a pseudo-differential operator on M and $\widetilde{M}$ a regular cover of M with countable transformation group H. Consider an inclusion $H < G$ and form the regular cover $\overline{M} = G \times_H \widetilde{M}$ of M. Then for the lifts $\widetilde{D}$ of D to $\widetilde{M}$ and $\overline{D}$ of $\widetilde{D}$ to $\overline{M}$,*

$$\mathrm{Index}_H(\widetilde{D}) = \mathrm{Index}_G(\overline{D}).$$

Proof. It is enough to see that $S_{\overline{D}} \cong \mathrm{Ind}_H^G(S_{\widetilde{D}})$. Indeed, it is well-known (see [9]) that for a Hilbert H-module A one has

$$\dim_H(A) = \dim_G(\mathrm{Ind}_H^G(A)).$$

For R a fixed set of representatives for G/H, the map

$$\begin{array}{rcl} \varphi_R : \mathrm{Ind}_H^G(S_{\widetilde{D}}) & \to & S_{\overline{D}} \\ f & \mapsto & \{f(r)\}_{r \in R} \end{array}$$

is well-defined by H-equivariance of the elements of $S_{\widetilde{D}}$ and one checks that it defines a G-equivariant isometric bijection. Similarly for the adjoint operators. □

The following example is a particular case of the previous lemma.

Example 3.2. Let us look at the case $\widetilde{M} = M \times G$. A section $\widetilde{s} \in C_c^\infty(\widetilde{M}, \pi^*E)$ is an element $\widetilde{s} = \{s_g\}_{g \in G}$ where $s_g \in C^\infty(M, E)$ and $s_g = 0$ for all but finitely many g's. Note that $L^2(\widetilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M, E)$. Now

$$\widetilde{D}\widetilde{s} = \{Ds_g\}_{g \in G} \in C_c^\infty(\widetilde{M}, \pi^*F)$$

and hence $S_{\widetilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_{\mathbb{C}}(S_D)$. In this identification the projection P onto $S_{\widetilde{D}}$ becomes the identity in $M_d(\mathcal{N}(G))$ and thus

$$\dim_G(S_{\widetilde{D}}) = \sum_{i=1}^{d} \langle e, e \rangle = d = \dim_{\mathbb{C}}(S_D).$$

A similar argument for D^* shows that in this case not only the L^2-Index of $\widetilde{D}$ coincides with the Index of D, but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

4. On K-homology

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator D on the closed manifold M can also be used to define an element $[D] \in K_0(M)$, the K-homology of M, and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined homomorphism (cf. [4])

$$\operatorname{Index}: K_0(M) \to \mathbb{Z},$$

such that $\operatorname{Index}([D]) = \operatorname{Index}(D)$. On the other hand, the projection $\mathrm{pr}: M \to \{pt\}$ induces, after identifying $K_0(\{pt\})$ with $\mathbb{Z}$, a homomorphism

$$\mathrm{pr}_*: K_0(M) \to \mathbb{Z}, \qquad (*)$$

which, as explained in [4], satisfies

$$\mathrm{pr}_*([D]) = \operatorname{Index}([D]).$$

More generally (cf. [4]), for a not necessarily finite CW-complex X, every $x \in K_0(X)$ is of the form $f_*[D]$ for some $f: M \to X$, and $K_0(X)$ is obtained as a colimit over $K_0(M_\alpha)$, where the M_α form a directed system consisting of closed Riemannian manifolds (these homology groups $K_0(X)$ are naturally isomorphic to the ones defined using the Bott spectrum; sometimes, they are referred to as K-homology groups with *compact supports*). The index map from above extends to a homomorphism

$$\operatorname{Index}: K_0(X) \to \mathbb{Z},$$

such that $\operatorname{Index}(x) = \operatorname{Index}([D])$ if $x = f_*[D]$, with $f: M \to X$.

We now consider the case of $X = BG$, the classifying space of the discrete group G, and obtain thus for any $f: M \to BG$ a commutative diagram

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\text{Index}} & \mathbb{Z} \\ {\scriptstyle f_*}\downarrow & & \| \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbb{Z}. \end{array}$$

Note that $(*)$ from above implies the following naturality property for the index homomorphism.

Lemma 4.1. *For any homomorphism $\varphi: H \to G$ one has a commutative diagram*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}} & \mathbb{Z} \\ {\scriptstyle (B\varphi)_*}\downarrow & & \| \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbb{Z}. \end{array}$$

□

We now turn to the L^2-index of Section 2. It extends to a homomorphism

$$\mathrm{Index}_G : K_0(BG) \to \mathbb{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*(y)$ for some $y = [D] \in K_0(M)$, $f : M \to BG$, M a closed smooth manifold and D an elliptic operator on M. Let $\widetilde{D}$ be the lifted operator to $\widetilde{M}$, the G-covering space induced by $f : M \to BG$. Then put

$$\mathrm{Index}_G(x) := \mathrm{Index}_G(\widetilde{D}).$$

One checks that $\mathrm{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where τ denotes the composite of the assembly map $K_0(BG) \to K_0(C_r^*G)$ with the natural trace $K_0(C_r^*G) \to \mathbb{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

Lemma 4.2. *For $H < G$ the following diagram commutes*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\mathrm{Index}_H} & \mathbb{R} \\ \downarrow & & \| \\ K_0(BG) & \xrightarrow{\mathrm{Index}_G} & \mathbb{R}. \end{array}$$

□

Atiyah's L^2-Index Theorem 2.1 for a given G can now be expressed as the statement (as already observed in [10])

$$\mathrm{Index}_G = \mathrm{Index} : K_0(BG) \to \mathbb{R}.$$

5. Algebraic proof of Atiyah's L^2-index theorem

Recall that a group A is said to be *acyclic* if $H_*(BA, \mathbb{Z}) = 0$ for $* > 0$. For G a countable group, there exists an embedding $G \to A_G$ into a countable acyclic group A_G. There are many constructions of such a group A_G available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension ΣBA_G is contractible, and therefore the inclusion $\{e\} \to A_G$ induces an isomorphism

$$K_0(B\{e\}) \xrightarrow{\cong} K_0(BA_G).$$

Our strategy is as follows. We show that the Atiyah L^2-Index Theorem holds in the special case of acyclic groups, and finish the proof combining the above embedding of a group into an acyclic group.

Proof of Theorem 2.1. If a group A is acyclic, the equation $\mathrm{Index}_A = \mathrm{Index}$ follows from the diagram

$$\begin{array}{ccccc} K_0(BA) & \xrightarrow{\mathrm{Index}_A} & \mathbb{R} & \xleftarrow{\mathrm{Index}} & K_0(BA) \\ \cong\uparrow & & \uparrow & & \cong\uparrow \\ K_0(B\{e\}) & \underset{\cong}{\xrightarrow{\mathrm{Index}_{\{e\}}}} & \mathbb{Z} & \underset{\cong}{\xleftarrow{\mathrm{Index}}} & K_0(B\{e\}) \end{array}$$

because $\mathrm{Index}_{\{e\}} = \mathrm{Index}$ on the bottom line. For a general group G, consider an embedding into an acyclic group A_G and complete the proof by using Lemma 3.1, together with Lemmas 4.1 and 4.2. □

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