Hattori-Stallings Trace and Euler Characteristics for Groups *

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Introduction

For G a group and P a finitely generated projective module over the integral group ring, Bass conjectured in [2] that the Hattori-Stallings rank of P should vanish on elements different from $1 \in G$, and proved it in many cases such as torsion-free linear groups. Later, this conjecture has been proved for many more groups, notably by Eckmann [13], Emmanouil [15] and Linnell [22]. The latest advances are given in [3] and the first section of the present paper is a quick survey of the Bass conjecture, together with an outline of the proof of the main result of [3].

In most cases, one proves a stronger conjecture, which asserts that the Hattori-Stallings rank of a finitely generated projective module over the complex group ring should vanish on elements of infinite order (that this conjecture is indeed stronger follows from Linnell's work [22]). Given a group G of type FP over $\mathbb{C}$, its complete Euler characteristic $E(G)$ is the Hattori-Stallings rank of an alternating sum of finitely generated projective modules over $\mathbb{C}G$, and on the elements of finite order, one could then ask of what the values do depend. It is Brown in [7] who first studied that question, proving a formula in many cases. In Section 2 we shall explain the basics to understand Brown's formula and propose Conjecture 1 below as a generalization. Our generalization amounts to putting Brown's work in the context of L^2-homology (not available at the time where [7] has been written), and applies to cases where Brown's formula is not available.

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Conjecture 1 *Let G be a group of type FP over $\mathbb{C}$ such that the centralizer of every element of finite order in G has finite L^2-Betti numbers. Then for every $s \in G$*

$$E(G)(s) = \chi^{(2)}(C_G(s)), \tag{1}$$

where $E(G)(s)$ is the s-component of the complete Euler characteristic of G and $\chi^{(2)}(C_G(s))$ is the L^2-Euler characteristic of the centralizer of s in G.

This formula, as opposed to the Bass conjecture, has nice stability properties that we discuss in Section 3. We describe in Section 4 a class of groups containing all G with cocompact $\underline{E}G$ for which Formula (1) holds. It is straightforward (see Lemma 2.1 below) that Formula (1) always holds for $s = 1$. If we write $\chi(G)$ for the naive Euler characteristic $\sum(-1)^i \dim_{\mathbb{C}} H_i(G;\mathbb{C})$ then for G satisfying Conjecture 1, we find (cf. Corollary 4.5)

$$\chi(G) = \sum_{[s]\in[G]} \chi^{(2)}(C_G(s)). \tag{2}$$

If $K(G,1)$ is a finite complex, then G satisfies Conjecture 1 and G is necessarily torsion-free so that Formula (2) reduces to Atiyah's celebrated theorem: $\chi(G) = \chi^{(2)}(G)$.

1 Review of the Bass conjecture

For a group G we denote by $\mathrm{HS} : K_0(\mathbb{C}G) \to HH_0(\mathbb{C}G) = \bigoplus_{[G]} \mathbb{C}$ the *Hattori-Stallings Trace*; $[G]$ stands for the set of conjugacy classes of G. If P denotes a finitely generated projective $\mathbb{C}G$-module and $[P] \in K_0(\mathbb{C}G)$ the corresponding element, we write

$$\mathrm{HS}(P) := \mathrm{HS}([P]) = \sum_{[s]\in[G]} \mathrm{HS}(P)(s)\cdot[s] \in \bigoplus_{[G]} \mathbb{C}$$

with $\mathrm{HS}(P)(s)$ depending on the conjugacy class $[s]$ of $s \in G$ only. Therefore, we can think of $\mathrm{HS}(P) : G \to \mathbb{C}$ as a class function. It is well-known that for $s \in G$ a *central* element of infinite order, one has $\mathrm{HS}(P)(s) = 0$. More generally, if $C_G(s)$ denotes the centralizer of $s \in G$, and $[G : C_G(s)]$ is finite and s has infinite order, then $\mathrm{HS}(P)(s) = 0$, because $\mathrm{HS}(P|_{C_G(s)})(s) = \mathrm{HS}(P)(s)$ (in general, if $H < G$ is a subgroup of finite index and $s \in H$, then

$\mathrm{HS}(P|_H)(s) = [C_G(s) : C_H(s)]\,\mathrm{HS}(P)(s)$, see [2, Corollary 6.3] or Chiswell's notes [11]).

Another very useful result in this context goes back to Bass (cf. [2], Proposition 9.2), and states that if $\mathrm{HS}(P)(s) \neq 0$ then there is an $N > 0$ such that for almost all primes p, the elements s^{p^N} are conjugate to s; note that in case s has infinite order, this implies that for almost all primes p, s is contained in a subgroup of G which is isomorphic to $\mathbb{Z}[1/p]$. According to Bass in [2], the following general vanishing theorem ought to be true:

Conjecture 2 (Bass Conjecture over $\mathbb{C}$) *For P a finitely generated projective $\mathbb{C}G$-module and $s \in G$ an element of infinite order,* $\mathrm{HS}(P)(s) = 0$.

The Bass Conjecture over $\mathbb{C}$ is known to hold for many groups, including:

- linear groups (cf. Bass [2]), which includes Braid Groups (they are linear: [19] and [5])

- groups with $\mathrm{cd}_{\mathbb{C}} \leq 2$ (cf. Eckmann [13]; see also Emmanouil [15], as well as [16] for more general results using techniques of cyclic homology), which includes one-relator groups and knot groups

- subgroups of semihyperbolic groups (this follows from results of Alonso and Bridson [1], see also [14] or the discussion in [24], following Corollary 7.17 of Part 1; for the definition of semihyperbolic groups the reader is referred to [6]); these include (subgroups of) word hyperbolic groups and cocompact CAT(0)-groups

- mapping class groups Γ_g of closed surfaces of genus g (cf. Corollary 7.17 (Part 1) of [24])

- amenable groups, more generally groups satisfying the Bost Conjecture (for the Bost Conjecture, see [20] and [26]) have been shown to satisfy the Bass Conjecture over $\mathbb{C}$ in [3]. For instance groups which have the *Haagerup Property* (also called *a-T-menable groups*). We recall that a group is said to have the Haagerup Property if it admits an isometric, metrically proper affine action on some Hilbert space (for a discussion of such groups, see [10]); the class of groups having the Haagerup property contains all countable groups which are extensions of amenable groups with free kernel, and is closed under subgroups, finite products, passing to the fundamental group of a countable, locally finite graph of groups with finite edge stabilizers (vertex stabilizers are assumed to have the Haagerup property), countable increasing unions, amalgamations $A *_B C$ with A and C both countable amenable and B central in A and C (use Propositions 4.2.12 and 6.2.3 of [10]) and passing to finite index supergroups. Groups which act metrically properly and isometrically on a uniformly locally finite, weakly δ-geodesic and strongly δ-bolic space (see [18] and [20]); examples of groups satisfying these conditions are word hyperbolic groups (see [25]) and cocompact CAT(0)-groups.

A prominent class of groups for which Conjecture 2 is not known in general, is the class of profinite groups. However, if G is any group and Q a finitely generated projective $\mathbb{Z}G$-module, then according to Linnell [22], if $s \neq 1$ is such that $\mathrm{HS}(\mathbb{C}G \otimes_{\mathbb{Z}G} Q)(s) \neq 0$, then s is contained in a subgroup of G isomorphic to the additive group $\mathbb{Q}$ of rationals. This in particular implies that for G profinite one has $\mathrm{HS}(\mathbb{C}G \otimes_{\mathbb{Z}G} Q)(s) = 0$ for all $s \in G \setminus \{1\}$, because $\mathbb{Q}$ cannot be a subgroup of a profinite group.

We give an outline of the strategy for proving the main result of [3], which states that the Bost Conjecture implies the Bass Conjecture over $\mathbb{C}$ (see Theorem 1.1 below). The Bost Conjecture asserts that the *Bost assembly map*

$$\beta_0^G : K_0^G(\underline{E}G) \to K_0(\ell^1 G)$$

is an isomorphism (see [20] and [26]). Here, the left hand side denotes the equivariant K-homology of the classifying space for proper actions of G, and the right hand side is the projective class group of the Banach algebra $\ell^1 G$ of summable complex valued functions on G.

Theorem 1.1 *Suppose that G satisfies the Bost Conjecture. Then G satisfies the Bass Conjecture over $\mathbb{C}$.*

Before outlining the proof of Theorem 1.1, we need to address some auxiliary constructions. We can extend $\mathrm{HS} : K_0(\mathbb{C}G) \to \bigoplus_{[G]} \mathbb{C}$ to a trace $\mathrm{HS}^1 : K_0(\ell^1 G) \to \prod_{[G]} \mathbb{C}$ as follows. If $[Q] \in K_0(\ell^1 G)$, with Q a finitely generated projective $\ell^1 G$-module, we choose an idempotent (n,n)-matrix $M = (m_{ij})$ with entries in $\ell^1 G$ representing Q (i.e. $(\ell^1 G)^n \cdot M \cong Q$ as left $\ell^1 G$-modules), then we put

$$\mathrm{HS}^1(Q) := \mathrm{HS}^1([Q]) := \{\sum_{i=1}^{n} \sum_{t \in [s]} m_{ii}(t)\}_{[s] \in [G]} \in \prod_{[G]} \mathbb{C}.$$

The $m_{ii}(t)$'s stand for the t-coefficients of $m_{ii} \in \ell^1 G$, $1 \leq i \leq n$. We will write $\mathrm{HS}^1(x)(s)$ for the $[s]$-component of $\mathrm{HS}^1(x)$, $x \in K_0(\ell^1 G)$. One checks that HS^1 is well-defined and compatible with the ususal Hattori-Stallings trace. To get informations on HS^1 via the Bost assembly map, we embed G into an acyclic group of a very special kind. Recall that a group G is called *acyclic*, if $H_i(G;\mathbb{Z}) = 0$ for $i > 0$. As proved in [3], every group G admits a functorial embedding into an acyclic group $A = A(G)$, which we call the *pervasively acyclic hull of* G, satisfying the following:

- For every finitely generated abelian subgroup $B < A$ the centralizer $C_A(B)$ is acyclic (such a group is called *pervasively acyclic*)
- A is countable if G is and the induced map on conjugacy classes $[G] \to [A]$ is injective.

In this context, the important feature of a pervasively acyclic group A is that its classifying space for proper actions is $K_0^A \otimes \mathbb{Q}$-*discrete*, meaning that the inclusion $\underline{E}A^0 \to \underline{E}A$ of the 0-skeleton induces a surjective map

$$K_0^A(\underline{E}A^0) \otimes \mathbb{Q} \to K_0^A(\underline{E}A) \otimes \mathbb{Q},$$

see Corollary 3.9 of [3]. In other words, all the information of the A-CW-complex $\underline{E}A$ captured by the equivariant K-homology is contained in its 0-skeleton $\underline{E}A^0 = \coprod_\alpha A/A_\alpha$, where $A_\alpha < A$ stands for a finite subgroup, corresponding to the stabilizer of some 0-cell of $\underline{E}A$. The equivariant K-homology we use here is the one defined by Davis and Lück (cf. [12]), arising from a spectrum over the orbit category of G. It is defined on the category of *all* G-CW-complexes; on proper, cocompact G-CW-complexes, this *representable* equivariant K-homology agrees with the one used in the original version of the Baum-Connes or Bost conjectures (see [17]) so that if X is a proper, not necessarily cocompact G-CW-complex, then $K_*^G(X) = \mathrm{colim}_{Y \subset X, Y/G\,\mathrm{compact}} K_*^G(Y)$, in accordance with the classical setup for the Baum-Connes and Bost conjectures. It follows that K_0^G is fully additive so that

$$K_0^A(\underline{E}A^0) = \bigoplus_\alpha K_0^A(A/A_\alpha) = \bigoplus_\alpha K_0^{A_\alpha}(\{pt\}) = \bigoplus_\alpha K_0(\ell^1 A_\alpha)$$

and $K_0(\ell^1 A_\alpha) \cong R_{\mathbb{C}}(A_\alpha)$, the additive group of the complex representation ring of the finite group A_α.

Outline of the proof of Theorem 1.1. Let P be a finitely generated projective $\mathbb{C}G$-module and assume that G satisfies the Bost Conjecture. Then $x := [\ell^1 G \otimes_{\mathbb{C}G} P]$ lies in the image of the Bost assembly map β_0^G and $\mathrm{HS}^1(x)$ captures the information contained in $\mathrm{HS}(P)$. We embed G into its pervasively acyclic hull $A(G) =: A$ and together with the standard embedding $\underline{E}A^0 \to \underline{E}A$ of the 0-skeleton this yields a commutative diagram

$$\begin{array}{ccccc}
 & & [P] \in K_0(\mathbb{C}G) & \xrightarrow{\mathrm{HS}} & \bigoplus_{[G]} \mathbb{C} \\
 & & \downarrow & & \downarrow \\
K_0^G(\underline{E}G) & \underset{\cong}{\xrightarrow{\beta_0^G}} & K_0(\ell^1 G) & \xrightarrow{\mathrm{HS}^1} & \prod_{[G]} \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
K_0^A(\underline{E}A) & \xrightarrow{\beta_0^A} & K_0(\ell^1 A) & \xrightarrow{\mathrm{HS}^1} & \prod_{[A]} \mathbb{C} \\
\uparrow & & \uparrow & & \uparrow \\
K_0^A(\underline{E}A^0) & \xrightarrow{\bigoplus \beta_0^{A_\alpha}} & \bigoplus_\alpha K_0(\ell^1 A_\alpha) & \xrightarrow{\bigoplus_\alpha \mathrm{HS}} & \bigoplus_{\alpha,[A_\alpha]} \mathbb{C}.
\end{array}$$

Using the facts that $K_0^A(\underline{E}A^0) \to K_0^A(\underline{E}A)$ is rationally surjective (since $\underline{E}A$ is $K_0^A \otimes \mathbb{Q}$-discrete), and that the induced map $\prod_{[G]} \mathbb{C} \to \prod_{[A]} \mathbb{C}$ is injective, we conclude by diagram chasing that $\mathrm{HS}^1(x)$ lies in the subspace of functions $[G] \to \mathbb{C}$, whose support is contained in the subset of those conjugacy classes of G, which are represented by elements of finite order. Therefore $\mathrm{HS}^1(x)(s) = 0$ for $s \in G$ of infinite order, which implies that $\mathrm{HS}(P)(s) = 0$ too, establishing the Bass conjecture over $\mathbb{C}$ for the group G.
QED

2 Euler characteristics

In this section we shall explain the basics to discuss Conjecture 1. Let G be a group of type FP over $\mathbb{C}$, meaning that there exists a resolution

$$P_* : 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{C}$$

with each P_i finitely generated projective over $\mathbb{C}G$; in case the P_i's may be chosen to be finitely generated and free over $\mathbb{C}G$, the G is termed of type

FF over $\mathbb{C}$. The element $W(G) := \sum_i (-1)^i [P_i] \in K_0(\mathbb{C}G)$ depends on G only and we call it the *Wall element*. Under the Hattori-Stallings trace, the Wall element $W(G)$ is mapped to $E(G) = \sum_{[s]\in[G]} E(G)(s)[s]$, the sum being taken over the set $[G]$ of conjugacy classes $[s]$ of elements $s \in G$. This is the *complete Euler characteristic of* G (see [27]). If G has a cocompact $\underline{E}G$, Conjecture 1 is true as it reduces to Brown's formula [7] that we shall now discuss. For G of type FP over $\mathbb{C}$, the *Euler characteristic of* G (in the sense of Bass [2] and Chiswell [11]) is given by $e(G) = E(G)(1)$. Note also that $W(G) = 0$ if and only if $e(G) = 0$ and G is of type FF over $\mathbb{C}$. Brown conjectures under suitable finiteness conditions for G the following formula:

$$E(G)(s) = \begin{cases} e(C_G(s)) & \text{if } s \text{ has finite order} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and proves it in many cases, including groups with cocompact $\underline{E}(G)$. Brown's assumptions always require in particular $C_G(s)$ to be of type FP over $\mathbb{C}$, and in this case we will show that Formula (1) reduces to Brown's formula (3). To do this, we first recall the definition of L^2-Euler characteristic. For $i \in \mathbb{N}$, the i-th L^2-Betti number is defined as the von Neumann dimension of the $\mathcal{N}(G)$-module $H_i\,(G;\mathcal{N}(G))$

$$\beta_i(G) = \dim_G H_i\,(G;\mathcal{N}(G)) \in [0,\infty],$$

where $\mathcal{N}(G)$ is the group von Neumann algebra of G (see Lück's book [23]). If $\sum(-1)^i\beta_i(G)$ converges, the L^2*-Euler characteristic* is defined as

$$\chi^{(2)}(G) = \sum_{i\in\mathbb{N}} (-1)^i \beta_i(G) \in \mathbb{R}. \tag{4}$$

In case G is finite, $\chi^{(2)}(C_G(s)) = 1/|C_G(s)|$ and Formulae (1) and (2) reduce to well-known results. With no finiteness restrictions imposed on G, one can find for any $r \in \mathbb{R}$ a group G with $\chi^{(2)}(G) = r$. However, if G is of type FP over $\mathbb{C}$ then $\chi^{(2)}(G) \in \mathbb{Q}$, as shown by the following.

Lemma 2.1 *Suppose that a group G is of type FP over $\mathbb{C}$. Then $\chi^{(2)}(G) = e(G)$ and $e(G)$ is a rational number.*

Proof. Let $P_* : 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{C}$ be a projective resolution of type FP for G. Then

$$\begin{aligned} \chi^{(2)}(G) &= \sum_{i\in\mathbb{N}} (-1)^i \beta_i(G) = \sum_{i\in\mathbb{N}} (-1)^i \dim_G \mathcal{N}(G) \otimes_{\mathbb{C}G} P_i \\ &= \sum_{i\in\mathbb{N}} (-1)^i HS(P_i)(1) = e(G). \end{aligned}$$

Here we used the fact that for a finitely generated projective $\mathbb{C}G$-module P, $\dim_G \mathcal{N}(G) \otimes_{\mathbb{C}G} P = HS(P)(1)$, which is actually just the Kaplansky trace of P; the Kaplansky trace of a finitely generated projective $\mathbb{C}G$-module is a rational number, by Zalesskii's theorem (see [8]). QED

The L^2-Betti numbers turn out to be often 0. In particular we mention the following vanishing result.

Theorem 2.2 (Cheeger-Gromov, [9]) *If G contains an infinite normal amenable subgroup, then $\beta_i(G) = 0$ for all $i \in \mathbb{N}$, and therefore $\chi^{(2)}(G) = 0$.*

This theorem immediately implies that for an arbitrary group G, the L^2-Euler characteristic of $C_G(s)$ is 0 for all $s \in G$ of infinite order, so that one more evidence for Conjecture 1 is the following simple observation: *If the Bass Conjecture over $\mathbb{C}$ holds for G, then Formula (1) holds on elements of infinite order.* Indeed, the Bass conjecture will say that the left hand side vanishes on elements of infinite order. The following fact on L^2-Euler characteristics will be used later, mainly for the case of subgroups $H < G$, with G of type FP over $\mathbb{C}$. Since then $\operatorname{cd}_{\mathbb{C}} H < \infty$, the Euler characteristic $\chi^{(2)}(H)$ is well-defined if and only if all L^2-Betti numbers $\beta_i(H)$ are finite.

Lemma 2.3 *Let H and K be groups with $\sum_i \beta_i(H)$ and $\sum_i \beta_i(K)$ convergent. Then $\chi^{(2)}(H \times K) = \chi^{(2)}(H)\chi^{(2)}(K)$.*

Proof. One uses the Künneth Formula for L^2-Betti numbers [9]: $\beta_n(H\times K) = \sum_{i+j=n} \beta_i(H)\beta_j(K)$, and takes the alternating sum; note that $\sum_n \beta_n(H\times K)$ is convergent so that $\chi^{(2)}(H \times K)$ is well-defined. QED

A 1-dimensional contractible G-CW-complex T with vertex set V and edge set E (for short: a G-tree) is given by a G-push-out

$$\begin{array}{ccc} (\coprod_{\beta\in E/G} G/G_\beta) \times S^0 & \longrightarrow & \coprod_{\alpha\in V/G} G/G_\alpha \\ \downarrow & & \downarrow \\ (\coprod_{\beta\in E/G} G/G_\beta) \times D^1 & \longrightarrow & T \end{array}$$

and the cellular chain complex of T has the form

$$0 \to \bigoplus_{\beta\in E/G} \mathbb{C}[G/G_\beta] \to \bigoplus_{\alpha\in V/G} \mathbb{C}[G/G_\alpha] \to \mathbb{C}.$$

The group G is then the fundamental group of a graph of groups $\{G_\gamma\}_{\gamma\in I}$, $I = V/G \sqcup E/G$; the graph is called *finite*, if I is a finite set (i.e. if the action of G on T is cocompact). If X is an arbitrary G-CW-complex, we write $H_*(X;\mathcal{N}(G)) := H_*(\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*^{\mathrm{cell}}(X))$ for its L^2-homology so that $H_*(G;\mathcal{N}(G)) = H_*(EG;\mathcal{N}(G))$.

Lemma 2.4 *Let G be the fundamental group of a (not necessarily finite) graph of groups $\{G_\gamma\}_{\gamma\in I}$, where $I = V/G \sqcup E/G$. If for each of the groups G_γ the series $\sum_i \beta_i(G_\gamma)$ is convergent and equals 0 for almost all $\gamma \in I$, then*

$$\chi^{(2)}(G) = \sum_{\alpha\in V/G} \chi^{(2)}(G_\alpha) - \sum_{\beta\in E/G} \chi^{(2)}(G_\beta).$$

Proof. The group G acts on a tree $T = (V, E)$ with chain complex

$$0 \to \bigoplus_{\beta\in E/G} \mathbb{C}[G/G_\beta] \to \bigoplus_{\alpha\in V/G} \mathbb{C}[G/G_\alpha] \to \mathbb{C}.$$

Take a projective resolution of this complex in the category of chain complexes over $\mathbb{C}G$, say $P_* \to Q_* \to R_*$, with P_* a projective resolution for $\bigoplus \mathbb{C}[G/G_\beta]$, Q_* one for $\bigoplus \mathbb{C}[G/G_\alpha]$ and R_* for $\mathbb{C}$. Upon tensoring with $\mathcal{N}(G)\otimes_{\mathbb{C}G} -$ we obtain a short exact sequence of chain complexes

$$\mathcal{N}(G)\otimes_{\mathbb{C}G} P_* \to \mathcal{N}(G)\otimes_{\mathbb{C}G} Q_* \to \mathcal{N}(G)\otimes_{\mathbb{C}G} R_*;$$

the exactness results from the fact that the sequences $P_i \to Q_i \to R_i$ are split exact for all i, because R_i is projective. Taking homology yields a long exact sequence of L^2-homology groups

$$\cdots \to H_{i+1}(G;\mathcal{N}(G)) \to \bigoplus_{\beta\in E/G} H_i(\mathrm{Ind}_{G_\beta}^G EG_\beta;\mathcal{N}(G)) \to$$

$$\bigoplus_{\alpha\in V/G} H_i(\mathrm{Ind}_{G_\alpha}^G EG_\alpha;\mathcal{N}(G)) \to H_i(G;\mathcal{N}(G)) \to \cdots .$$

We used here that a $\mathbb{C}G$-projective resolution of $\mathbb{C}[G/G_\gamma]$ is chain homotopy equivalent to the cellular $\mathbb{C}G$-chain complex of the induced G-CW-complex $\mathrm{Ind}_{G_\gamma}^G EG_\gamma = G\times_{G_\gamma} EG_\gamma$. Therefore

$$H_*(\mathcal{N}(G)\otimes_{\mathbb{C}G} P_*) = \bigoplus_{\beta\in E/G} H_*(\mathrm{Ind}_{G_\beta}^G EG_\beta;\mathcal{N}(G))$$

and similarly for $H_*(\mathcal{N}(G) \otimes_{\mathbb{C}G} Q_*)$. According to [23] (Theorem 6.54, (7)), for any induced G-CW-complex $\mathrm{Ind}_{G_\gamma}^G X$ one has

$$\dim_G H_i(\mathrm{Ind}_{G_\gamma}^G X; \mathcal{N}(G)) = \dim_{G_\gamma} H_i(X; \mathcal{N}(G_\gamma))$$

and it follows that $\dim_G H_i(\mathrm{Ind}_{G_\gamma}^G EG_\gamma; \mathcal{N}(G)) = \beta_i(G_\gamma)$. Thus, by taking the alternating sum of L^2-Betti numbers in the long exact homology sequence above, the desired formula follows. QED

There are groups G of type FP over $\mathbb{C}$ containing centralizers $C_G(s)$ which are not of type FP over $\mathbb{C}$. Such examples have first been constructed by Leary and Nucinkis in [21], and those cannot satisfy Brown's formula, because then $e(C_G(s))$ is not defined. The following group G is a simple example for which Formula (1) holds whereas (3) doesn't apply. Take first a group $\mathcal{G}$ as described by Leary-Nucinkis in [21] with the following property:

$\mathcal{G}$ is of type FP over $\mathbb{C}$ and contains an element $t \in \mathcal{G}$ of finite order such that $C_{\mathcal{G}}(t)$ is not of type FP over $\mathbb{C}$.

Then the right hand side of Brown's formula (3) doesn't make sense for the group $G = \mathcal{G} \times \mathbb{Z}$, which is of type FP over $\mathbb{C}$ but none of the centralizers $C_G((t,n)) = C_{\mathcal{G}}(t) \times \mathbb{Z}$ are; note that $(t,n) \in G$ is of finite order if and only if $n = 0$. But nevertheless, the group G satisfies Conjecture 1 because of the following.

Lemma 2.5 *Let H be a group of type FP over $\mathbb{C}$ and $G := H \times \mathbb{Z}$. Then G is of type FF over $\mathbb{C}$, satisfies Conjecture 1 and $W(G) = 0 \in K_0(\mathbb{C}G)$.*

Proof. Let $P_* : 0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{C} \to 0$ be a resolutions of type FP over $\mathbb{C}$ for H and $D_* : 0 \to \mathbb{C}\langle z \rangle \to \mathbb{C}\langle z \rangle \to \mathbb{C} \to 0$ be the projective resolution for $\mathbb{Z} = \langle z \rangle$ with the map $\mathbb{C}\langle z \rangle \to \mathbb{C}\langle z \rangle$ given by multiplication with $1 - z$. Then $E_* = P_* \otimes D_* \to \mathbb{C} \to 0$ is a resolution of type FP over $\mathbb{C}$ for $G = H \times \mathbb{Z}$, and since $E_i = (P_i \otimes \mathbb{C}\langle z \rangle) \oplus (P_{i-1} \otimes \mathbb{C}\langle z \rangle)$, we see that $W(G) = \sum_{i=0}^{n+1} (-1)^i [E_i] = 0$ (terms cancel pairwise); hence G is of type FF over $\mathbb{C}$ and $E(G) = 0$ so that $E(G)(s) = 0$ for every $s \in G$. On the other hand, the centralizer of $s = (u,v) \in H \times \mathbb{Z}$ contains the normal subgroup $\{1_H\} \times \mathbb{Z}$ so that $\chi^{(2)}(C_G(s)) = 0$ as well. QED.

It follows that Conjecture 1 holds for any group of type FP over $\mathbb{C}$ of the form $H \times \mathbb{Z}$, because both sides are zero; we shall construct non-zero examples later (recall that if $H \times \mathbb{Z}$ is of type FP over $\mathbb{C}$ then so is H by

Proposition 2.7 of [4]). We we will show in Section 4 (Theorem 4.6) that for each $\rho \in \mathbb{Q}$ there exists a group $G(\rho)$ of type FP over $\mathbb{C}$ containing an element s of finite order such that $C_{G(\rho)}(s)$ is not of type FP over $\mathbb{C}$ but such that $G(\rho)$ nevertheless satisfies Conjecture 1, with $E(G(\rho))(s) = \rho$.

3 Stability properties of Formula (1)

In this section we shall study some stability properties of Formula (1), starting with the following.

Lemma 3.1 *Let A and B be two groups of type FP over $\mathbb{C}$ such that A satisfies Formula (1) for some $a \in A$, and B satisfies it for some $b \in B$. Then $G = A \times B$ satisfies Formula (1) for the element (a, b).*

Proof. Let $P_* : 0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{C} \to 0$ be a resolution of type FP over $\mathbb{C}$ for A and $Q_* : 0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to \mathbb{C} \to 0$ one for B (by adding trivial modules we can assume that both resolutions have the same length). A projective resolution of type FP over $\mathbb{C}$ for $G = A \times B$ is given by $E_* = P_* \otimes Q_* \to \mathbb{C} \to 0$. For an element $s = (a, b) \in G$ we compute

$$\begin{aligned}
HS(W(G))(s) &= \sum_{i=0}^{2n}(-1)^i HS(E_i)(s) = \sum_{i=0}^{2n}(-1)^i \sum_{k+\ell=i} HS(P_k \otimes Q_\ell)(a,b) \\
&= \sum_{i=0}^{2n} \sum_{k+\ell=i} (-1)^{k+\ell} HS(P_k)(a) HS(Q_\ell)(b) \\
&= HS(W(A))(a) HS(W(B))(b) = \chi^{(2)}(C_A(a))\chi^{(2)}(C_B(b)) \\
&= \chi^{(2)}(C_A(a) \times C_B(b)).
\end{aligned}$$

Here we used in the second line the fact that for P and Q finitely generated projective modules over $\mathbb{C}A$ and $\mathbb{C}B$ respectively, $HS(P \otimes Q)(a, b) = HS(P)(a)HS(Q)(b)$. We conclude using Lemma 2.3 and the fact that $C_G(a, b) = C_A(a) \times C_B(b)$. Note that the L^2-Betti numbers of $C_A(a)$ are finite, and trivial for large degrees, because $C_A(a)$ is assumed to have a well-defined L^2-Euler characteristic and $\mathrm{cd}_{\mathbb{C}}\, C_A(a)$ is finite; a similar remark applies to $C_B(b)$. QED

Definition 3.2 (Condition (F)) *The fundamental group G of a finite graph of groups $\{G_\gamma\}$ satisfies* Condition (F), *if the G-action on the associated standard tree T is such that for every element of finite order $s \in G$, the action of $C_G(s)$ on the fixed tree T^s satisfies the hypothesis of Lemma 2.4.*

Remark 3.3 *Condition (F) amounts to say that for any element of finite order $s \in G$ and for each of the stabilizers $H < C_G(s)$ appearing on the fixed tree T^s, the series $\sum_i \beta_i(H)$ is convergent and equals 0 for all but finitely many conjugacy classes (H).*

Lemma 3.4 *Let G be the fundamental group of a finite graph of groups.*

(i) If all edge and vertex groups satisfy Formula (1) at all elements of infinite order, then so does G.

(ii) If G satisfies Condition (F) and all edge and vertex groups satisfy Formula (1) at all elements of finite order, then G satisfies Formula (1) at all elements of finite order.

Proof. The group G acts cocompactly on a tree $T = (V, E)$, yielding a resolution $0 \to \bigoplus_{\beta \in E/G} \mathbb{C}[G/G_\beta] \to \bigoplus_{\alpha \in V/G} \mathbb{C}[G/G_\alpha] \to \mathbb{C} \to 0$. Each of the groups G_γ (for $\gamma \in V/G \sqcup E/G$) is of type FP over $\mathbb{C}$ (by assumption), so let us denote by $P_*^\gamma : 0 \to P_n^\gamma \to \cdots \to P_0^\gamma \to \mathbb{C} \to 0$ a corresponding resolution of type FP. Tensoring by $\mathbb{C}[G/G_\gamma]$ yields the following resolutions of type FP over $\mathbb{C}$ of induced modules: $\tilde{P}_*^\gamma : 0 \to \tilde{P}_n^\gamma \to \cdots \to \tilde{P}_1^\gamma \to \tilde{P}_0^\gamma \to \mathbb{C}[G/G_\gamma] \to 0,$ for $\gamma \in V/G \sqcup E/G$, so that the Wall element for G is given by

$$W(G) = \sum_{\alpha \in V/G} [\mathbb{C}[G/G_\alpha]] - \sum_{\beta \in E/G} [\mathbb{C}[G/G_\beta]],$$

where $[\mathbb{C}[G/G_\gamma]] = \sum_{j=0}^n (-1)^j [\tilde{P}_j^\gamma] = i_*^\gamma W(G_\gamma) \in K_0(\mathbb{C}G)$. The complete Euler characteristic of G is then given by

$$E(G) = \sum_{\alpha \in V/G} i_*^\alpha E(G_\alpha) - \sum_{\beta \in E/G} i_*^\beta E(G_\beta).$$

(i) Now let us take $s \in G$ of infinite order. By Cheeger-Gromov's Theorem 2.2 of this note $\chi^{(2)}(C_G(s)) = 0$; on the other hand, $E(G)(s) = 0$ because $E(G_\gamma)(t) = 0$ for any $\gamma \in V/G \sqcup E/G$ and any t of infinite order, by assumption on the G_γ's.

(ii) If $s \in G$ has finite order, then

$$\begin{aligned} E(G)(s) &= \sum_{[x] \in [s, G_\alpha]} E(G_\alpha)(x) - \sum_{[y] \in [s, G_\beta]} E(G_\beta)(y) \\ &= \sum_{[x] \in [s, G_\alpha]} \chi^{(2)}(C_{G_\alpha}(x)) - \sum_{[y] \in [s, G_\beta]} \chi^{(2)}(C_{G_\beta}(y)) \end{aligned}$$

because by assumption the G_γ's satisfy Formula (1) at elements of finite order (we used here the notation $[s, G_\gamma]$ for the conjugacy classes of elements in G_γ which are G-conjugate to s). So to conclude we need to show that the last line of the above equation is equal to $\chi^{(2)}(C_G(s))$, which we will do now. We think of the G_γ's as representatives for the stabilizers of the G-action on the standard tree T of the given graph of groups so that a general stabilizer will have the form $tG_\gamma t^{-1}$. Since s has finite order, $T^s = (V^s, E^s)$ is a non-empty tree, upon which $C_G(s)$ acts via the restriction of the G-action on T. The stabilizer of a vertex or an edge $\in T^s$ has the form $C_G(s) \cap tG_\gamma t^{-1}$, where $s \in tG_\gamma t^{-1}$, so that $C_G(s) \cap tG_\gamma t^{-1} \cong C_{G_\gamma}(t^{-1}st)$. Moreover, by assumption G satisfies Condition (F), and hence $\chi^{(2)}(C_{G_\gamma}(tst^{-1}))$ is well-defined so that $\chi^{(2)}(C_G(s))$ is well defined too and, by Lemma 2.4 satisfies

$$\chi^{(2)}(C_G(s)) = \sum_{x \in I} \chi^{(2)}(C_{G_\alpha}(x)) - \sum_{y \in J} \chi^{(2)}(C_{G_\beta}(y))$$

with index set I corresponding to $V^s/C_G(s)$. But this set corresponds bijectively to conjugacy classes of elements x in the $[G_\alpha]$'s, which are G-conjugate to s; similarly for J. QED

4 Conjecture 1 and two classes of groups

To begin with, we consider the following class $\mathcal{B}_\infty$ of groups.

Definition 4.1 *Let $\mathcal{B}_\infty$ denote the smallest class of groups which contains all groups of type FF over $\mathbb{C}$, all groups of type FP over $\mathbb{C}$ which satisfy the Bass Conjecture over $\mathbb{C}$, all groups G with cocompact $\underline{E}G$, all groups $G = H \times \mathbb{Z}$ with H of type FP over $\mathbb{C}$ and which is closed under finite products of groups and under passing to the fundamental group of a finite graph of groups.*

Clearly all groups in $\mathcal{B}_\infty$ are of type FP over $\mathbb{C}$. In particular, the Wall element $W(G) \in K_0(\mathbb{C}G)$ is defined for all groups in $\mathcal{B}_\infty$. Examples of groups in $\mathcal{B}_\infty$ include word hyperbolic groups, braid groups, cocompact CAT(0)-groups, Coxeter groups, mapping class groups of surfaces, knot groups, finitely generated one-relator groups, S-arithmetic groups, Artin groups, amenable groups of type FP over $\mathbb{C}$. Many more groups can be obtained using the closure properties mentioned before; the groups thus obtained are in general not known to satisfy the Bass conjecture over $\mathbb{C}$. We do not know of any group of type FP over $\mathbb{C}$ not belonging to $\mathcal{B}_\infty$. As we have seen, there are groups G in $\mathcal{B}_\infty$ containing x of finite (resp. infinite) order, whose centralizer $C_G(x)$ is not of type FP over $\mathbb{C}$ and, therefore, $E(C_G(x))$ is not defined and $C_G(x) \notin \mathcal{B}_\infty$. But nevertheless, the following holds.

Theorem 4.2 *Let G be a group in $\mathcal{B}_\infty$ and $s \in G$ an element of infinite order. Then Formula (1) holds at s:*

$$E(G)(s) = 0 = \chi^{(2)}(C_G(s)).$$

Proof. We have already seen that the right hand side is 0 (cf. Cheeger-Gromov's Theorem 2.2 in this note). The left hand side is certainly 0 in case G is of type FF over $\mathbb{C}$ or if G satisfies the Bass Conjecture over $\mathbb{C}$ or if $\underline{E}G$ is cocompact. Moreover, by Lemmas 3.1 and 3.4 (i), if $G = H \times K$ or G is the fundamental group of a finite graph of groups G_α and if $E(L)(t) = 0$ for all t of infinite order in L, where L is one of the groups H, K, G_α, then $E(G)(s) = 0$ for all elements of infinite order $s \in G$. Finally, $G = H \times \mathbb{Z}$ certainly satisfies $E(G)(s) = 0$ for all s (see Lemma 2.5). QED

We now describe a class of groups $\mathcal{B} \subset \mathcal{B}_\infty$ containing many examples of groups G with $E(G)(s) \neq 0$ for some $s \neq e$ in G satisfying Conjecture 1, but such that the corresponding centralizer $C_G(s)$ is not of type FP over $\mathbb{C}$.

Definition 4.3 *Let $\mathcal{B}$ denote the smallest class of groups which contains all groups G with cocompact $\underline{E}G$, all groups $G = H \times \mathbb{Z}$ with H of type FP over $\mathbb{C}$ and which is closed under finite products of groups and under passing to the fundamental group of a finite graph of groups which satisfy Condition (F).*

Theorem 4.4 *The groups of the class $\mathcal{B}$ satisfy Conjecture 1.*

Proof. This follows by applying Lemmas 2.5, 3.1 and 3.4. QED

Corollary 4.5 *For G satisfying Conjecture 1, $\chi(G) = \sum_{[s]\in[G]} \chi^{(2)}(C_G(s))$.*

Proof. By definition we have that

$$\chi(G) = \sum_i (-1)^i \dim_{\mathbb{C}} H_i(G; \mathbb{C}) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}G} P_i,$$

where $P_* \to \mathbb{C}$ is a resolution of G of type FP over $\mathbb{C}$. It implies that $\sum_{[s]\in[G]} E(G)(s) = \chi(G)$, because for P a finitely generated projective $\mathbb{C}G$-module, $\sum_{[s]\in[G]} \mathrm{HS}(P)(s) = \dim_{\mathbb{C}} \mathbb{C} \otimes_{\mathbb{C}G} P$. The desired result now follows from Formula (1). QED

We shall now construct explicit non-trivial examples in the class $\mathcal{B}$. More precisely we prove the following.

Theorem 4.6 *Given $\rho \in \mathbb{Q}$ there exists a group $G = G(\rho)$ of type FP over $\mathbb{C}$ with $s \in G$ of order 2 such that G satisfies Conjecture 1, with*

$$E(G)(s) = \chi^{(2)}(C_G(s)) = \rho$$

but with the centralizer $C_G(s)$ not of type FP over $\mathbb{C}$.

Before proceeding with the proof we need the following.

Lemma 4.7 *For $\rho \in \mathbb{Q}$ there exist a group $G_\rho \in \mathcal{B}$ with $\chi^{(2)}(G_\rho) = \rho$.*

Proof. Since a free group F_n of rank n satisfies $\chi^{(2)}(F_n) = 1 - n$, one has for $n, k \geq 0$ that $\chi^{(2)}((F_2 \times F_{n+1}) * F_k) = n - k$, so that for $\ell > 0$

$$\chi^{(2)}(((F_2 \times F_{n+1}) * F_k) \times \mathbb{Z}/\ell\mathbb{Z}) = \frac{n-k}{\ell}.$$

The group $G = ((F_2 \times F_{n+1}) * F_k) \times \mathbb{Z}/\ell\mathbb{Z}$ admits a cocompact $\underline{E}G$ via its obvious quotient action on $E(G/(\mathbb{Z}/\ell\mathbb{Z}))$, with orbit space the finite complex $((\vee^2 S^1) \times (\vee^{n+1} S^1)) \vee (\vee^k S^1)$, thus $G \in \mathcal{B}$. QED

Proof of Theorem 4.6. Let $\mathcal{G}$ be one of the groups described in [21], Example 9, such that $\mathcal{G}$ is of type FP over $\mathbb{C}$, $s \in \mathcal{G}$ is an element of order 2 and $C_{\mathcal{G}}(s)$ is not finitely generated. By definition of $\mathcal{B}$, the group $H := \mathcal{G} \times \mathbb{Z}$ belongs to $\mathcal{B}$, and $C_H((s,0))$ is not finitely generated, because it maps onto $C_{\mathcal{G}}(s)$. Writing t for $(s,0)$, we form $K := H *_{\langle t \rangle} H \in \mathcal{B}$. Thus, K is the fundamental group of a finite graph of groups $\{H, \langle t \rangle\}$, with associated tree T. If $w \in K$ has finite order with w not conjugate to t, the edge stabilizers of the $C_K(w)$ action on T^w are all trivial, and the vertex stabilizers are isomorphic to $C_H(z)$ for some element z of order 2 in H, thus $\beta_i(C_H(z)) = 0$ for all i, because such a centralizer contains a normal infinite cyclic subgroup. The centralizer of $\langle t \rangle$ in K decomposes as a fundamental group of a graph of groups of the form $\{H_\delta, \langle t \rangle\}$ with the H_δ's again isomorphic to groups $C_H(w)$, $w \in H$ of order 2, so that $\beta_i(H_\delta) = 0$ for all i and all δ. It follows that K satisfies Condition (F) and $\chi^{(2)}(C_K(t)) = -\chi^{(2)}(\langle t \rangle) = -1/2$. Note that $C_K(t)/\langle t \rangle$ maps onto $C_H(t)/\langle t \rangle$, which shows that $C_K(t)$ is not finitely generated. Forming $G := K \times G_{-2\rho} \in \mathcal{B}$ where $G_{-2\rho}$ is obtained following

Lemma 4.7 above, gives a group with $C_G(t) = C_K(t) \times G_{-2\rho}$ not of type FP over $\mathbb{C}$ (because it is not finitely generated), but

$$\chi^{(2)}(C_G(t)) = \chi^{(2)}(C_K(t)) \cdot \chi^{(2)}(G_{-2\rho}) = -\frac{1}{2} \cdot (-2\rho) = \rho.$$

QED

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