EQUIVARIANT K-HOMOLOGY AND RESTRICTION TO FINITE CYCLIC SUBGROUPS

MICHEL MATTHEY AND GUIDO MISLIN

Abstract. For a discrete group G, we prove that a G-map between proper G-CW-complexes induces an isomorphism in G-equivariant K-homology if it induces an isomorphism in C-equivariant K-homology for every finite cyclic subgroup C of G. As an application, we show that the source of the Baum-Connes assembly map, namely $K_*^G(E(G,\mathcal{F}in))$, is isomorphic to $K_*^G(E(G,\mathcal{FC}))$, where $E(G,\mathcal{FC})$ denotes the classifying space for the family of finite cyclic subgroups of G. Letting $\mathcal{VC}$ be the family of virtually cyclic subgroups of G, we also establish that $K_*^G(E(G,\mathcal{F}in)) \cong K_*^G(E(G,\mathcal{VC}))$ and related results.

1. Introduction

The goal of this note is to prove the following result.

Theorem 1.1. *For a discrete group G, the natural map $E(G,\mathcal{FC}) \to E(G,\mathcal{F}in)$ of classifying spaces for the family of all finite, respectively all finite cyclic subgroups of G, induces an isomorphism $K_*^G\big(E(G,\mathcal{FC})\big) \xrightarrow{\cong} K_*^G\big(E(G,\mathcal{F}in)\big)$.*

The Baum-Connes Conjecture [3] can be viewed as the statement that the natural map $K_*^G\big(E(G,\mathcal{F}in)\big) \to K_*^G\big(E(G,\mathcal{A}\ell\ell)\big)$ is an isomorphism, where $\mathcal{A}\ell\ell$ stands for the family of all subgroups of G. As a consequence, we get a reformulation of it.

Corollary 1.2. *For a discrete group G, the following statements are equivalent:*

(i) *G satisfies the Baum-Connes conjecture;*

(ii) $K_*^G\big(E(G,\mathcal{FC})\big) \xrightarrow{\cong} K_*(C_r^*G)$.

In (ii) the homomorphism is induced by the constant map, using the standard identification of $K_*^G(pt) = K_*^G\big(E(G,\mathcal{A}\ell\ell)\big)$ with $K_*(C_r^*G)$.

The spaces $X = E(G,\mathcal{FC})$ and $Y = E(G,\mathcal{F}in)$ being C-contractible for every finite cyclic subgroup C of G, Theorem 1.1 will follow from the following general result concerning equivariant K-homology.

Theorem 1.3. *Let G be a discrete group and let $f\colon X \to Y$ be a G-map between proper G-CW-complexes. Consider the following properties:*

(i) *the induced map $K_*^C(f)\colon K_*^C(X) \to K_*^C(Y)$ is an isomorphism for every finite cyclic subgroup C of G;*

(ii) *the induced map $K_*^F(f)\colon K_*^F(X) \to K_*^F(Y)$ is an isomorphism for every finite subgroup F of G;*

(iii) *the induced map $K_*^G(f)\colon K_*^G(X) \to K_*^G(Y)$ is an isomorphism.*

Then, (i) and (ii) are equivalent and imply (iii).

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For finite groups, a *co*homological version of Theorem 1.3 goes back to Jackowski in [12]. Theorem 1.3 will follow from a reduction to the finite group case, together with an adaption of the tom Dieck Localization Theorem for equivariant homology [22] to the setting of equivariant K-homology. Actually, the result for K-cohomology requires a completion theorem (also proved in [12]), dual to tom Dieck's localization result.

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2. Recollection on equivariant homology theories

In this section, we review basic properties of equivariant homology theories. It is in this setting that we will state, in Section 3, the general form of the main results of the Introduction.

Let G be a (discrete) group. A G-equivariant homology theory $\mathcal{H}_*^G(-)$, with values in abelian groups, is a collection of functors $\mathcal{H}_n^G(-)\colon (X,A) \mapsto \mathcal{H}_n^G(X,A)$, with $n \in \mathbb{Z}$, from G-CW-pairs to abelian groups together with natural transformations $\partial_n^G(X,A)\colon\ \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\varnothing)$; we also write ∂_n^G for short. These are required: to be *G-homotopy invariant*, to fit in the *long exact sequence* of a pair and to satisfy *excision*. We moreover require $\mathcal{H}_*^G(-)$ to be *additive*, in the sense that it satisfies the *disjoint union axiom*, i.e. if $\{X_i\}_{i\in I}$ is a collection of G-CW-complexes, then there is an isomorphism

$$\bigoplus_{i\in I}\mathcal{H}_*^G(\mathrm{incl}_i)\colon\ \bigoplus_{i\in I}\mathcal{H}_*^G(X_i) \xrightarrow{\cong} \mathcal{H}_*^G\Big(\coprod_{i\in I} X_i\Big)\,.$$

For a pointed G-CW-complex X, with G acting trivially on the base-point x_0, as usual, we define $\widetilde{\mathcal{H}}_*^G(X)$ as the cokernel of the map induced by the inclusion of x_0.

For the sequel, we need a notation. Let $\varphi\colon H \to G$ be a group homomorphism, and X an H-CW-complex. We denote by $\mathrm{ind}_\varphi(X)$ the space $G \times X$ modulo the equivalence relation $(g,x) \sim (g\varphi(h), h^{-1}x)$ for all $g \in G$, $h \in H$ and $x \in X$. The class of (g,x) is denoted by $[g,x]$. The space $\mathrm{ind}_\varphi(X)$ is a G-CW-complex for the action on the 'first factor', i.e. $\gamma \cdot [g,x] := [\gamma g, x]$ for γ, $g \in G$ and $x \in X$. For an H-CW-pair, we denote $\big(\mathrm{ind}_\varphi(X), \mathrm{ind}_\varphi(A)\big)$ simply by $\mathrm{ind}_\varphi(X,A)$. When φ is an inclusion, we also write $\mathrm{ind}_H^G(X)$ and $G \times_H X$ for $\mathrm{ind}_\varphi(X)$ and similarly for a pair.

Let $\mathscr{G}rps$ and $\mathscr{F}in\mathscr{G}rps$ be the class of groups and finite groups respectively. An *equivariant homology theory* $\mathcal{H}_*^?(-)$ is a collection $\{\mathcal{H}_*^G(-)\}_{G\in\mathscr{G}rps}$ of G-equivariant homology theories equipped with an *induction structure*, that is, for a group homomorphism $\varphi\colon H \to G$ and an H-CW-pair (X,A) such that $\mathrm{Ker}(\varphi)$ acts freely on X, there is a natural isomorphism

$$\mathrm{ind}_\varphi\colon \mathcal{H}_*^H(X,A) \xrightarrow{\cong} \mathcal{H}_*^G\big(\mathrm{ind}_\varphi(X,A)\big)$$

satisfying the following properties: it is compatible with boundaries; it is compatible with conjugations (i.e. if c_g denotes the conjugation of G by the element $g \in G$, then $\mathrm{ind}_{c_g} = \mathcal{H}_*^G(\bar{h})$ holds, where $\bar{h}\colon (X,A) \xrightarrow{\cong} \mathrm{ind}_{c_g}(X,A)$, $x \mapsto (e, g^{-1}x)$ is the canonical G-homeomorphism). If φ is an inclusion, we also write ind_H^G for ind_φ.

Observe that for an equivariant homology theory $\mathcal{H}^?_*(-)$, a group G and a *free* G-CW-complex X, the projection $\pi\colon G \twoheadrightarrow \{e\}$ induces an isomorphism

$$\mathrm{ind}_\pi\colon \mathcal{H}^G_*(X) \xrightarrow{\cong} \mathcal{H}_*\big(\mathrm{ind}_\pi(X)\big) = \mathcal{H}_*(G\backslash X)\,.$$

A *restriction structure* on an equivariant homology theory $\mathcal{H}^?_*(-)$ is given by the following additional data. For an injective group homomorphism $\iota\colon H \hookrightarrow G$ with $\iota(H)$ of finite index in G, there is a natural transformation

$$\mathrm{res}_\iota\colon \mathcal{H}^G_*(X,A) \to \mathcal{H}^H_*\big(\mathrm{res}_\iota(X,A)\big)\,,$$

where $\mathrm{res}_\iota(X,A)$ merely denotes (X,A) viewed as an H-CW-pair via ι; if ι is an inclusion, we write res^G_H for res_ι. These are required: to be compatible with the boundary homomorphisms; to be functorial, i.e. if $\iota'\colon G \hookrightarrow \Gamma$ is a second monomorphism with $\iota'(G)$ of finite index in Γ, then $\mathrm{res}_{\iota'\circ\iota} = \mathrm{res}_\iota \circ \mathrm{res}_{\iota'}$; to be compatible with the induction structure, i.e. if $\varphi\colon H \xrightarrow{\cong} G$ is an isomorphism, then

$$\mathcal{H}^G_*(\tilde{h}) \circ \mathrm{ind}_\varphi \circ \mathrm{res}_\varphi = \mathrm{id}_{\mathcal{H}^G_*(X)}\,,$$

where $\tilde{h}\colon \mathrm{ind}_\varphi \mathrm{res}_\varphi(X) \xrightarrow{\cong} X$ is the canonical G-homeomorphism; to satisfy the *double coset formula*, i.e. for subgroups H and K of G with K of finite index in G, and for an H-CW-pair (X,A), the compositions

$$\mathrm{res}^G_K \circ \mathrm{ind}^G_H\colon \mathcal{H}^H_*(X,A) \to \mathcal{H}^K_*\big(\mathrm{res}^G_K \mathrm{ind}^G_H(X,A)\big)$$

and

$$\mathcal{H}^K_*(\hat{h}) \circ \Phi_* \circ \bigoplus_{KgH \in K\backslash G/H} (\mathrm{ind}_{\tilde{c}_g} \circ \mathrm{res}^H_{L_g})$$

coincide, where Φ_* is the isomorphism given by the finite disjoint union axiom, and for $g \in G$, $L_g := H \cap g^{-1}Kg$, $\tilde{c}_g\colon L_g \to K$ is the conjugation by g, and

$$\hat{h}\colon \coprod_{KgH \in K\backslash G/H} \mathrm{ind}_{\tilde{c}_g} \mathrm{res}^H_{L_g}(X,A) \xrightarrow{\cong} \mathrm{res}^G_K \mathrm{ind}^G_H(X,A)$$

is the canonical K-homeomorphism.

A G-equivariant homology theory $\mathcal{H}^G_*(-)$ is *multiplicative (with unit)* if for two G-CW-pairs (X,A) and (Y,B), and for $p,\, q \in \mathbb{Z}$, there is an external product

$$\times\colon \mathcal{H}^G_p(X,A) \otimes_{\mathbb{Z}} \mathcal{H}^G_q(Y,B) \to \mathcal{H}^G_{p+q}\big((X,A)\times(Y,B)\big)\,,$$

that is natural both in (X,A) and (Y,B). As usual, $(X,A)\times(Y,B)$ is the G-CW-pair $\big(X\times Y,\ (X\times B)\cup(A\times Y)\big)$. The external product is required to be compatible with the boundary homomorphisms in the sense that

$$\partial^G_{p+q}(\alpha\times\beta) = \partial^G_p(\alpha)\times\beta + (-1)^p\,\alpha\times\partial^G_q(\beta)\,,$$

for $\alpha \in \mathcal{H}^G_p(X,A)$ and $\beta \in \mathcal{H}^G_q(Y,B)$, and to posses a unit $1_{\mathcal{H}} = 1^G_{\mathcal{H}} \in \mathcal{H}^G_0(pt)$. For example, if an equivariant homology theory $\mathcal{H}^?_*(-)$ has a multiplicative structure in the strong sense of Lück [16], then $\mathcal{H}^G_*(-)$ is multiplicative with unit in our sense, for every group G.

Let F be a finite group. We denote by $\mathrm{Rep}(F)$ the class of all finite dimensional complex representations of F. For a locally finite (hence locally compact) F-CW-complex Z with F therefore acting via *proper* homeomorphisms, we designate its one-point compactification by $Z^\infty = Z \cup \{\infty\}$, with F acting trivially on the base-point ∞; this is (non-canonically) an F-CW-complex as well. An F-equivariant homology theory $\mathcal{H}_*^F(-)$ is called *stable* if given $V \in \operatorname{Rep}(F)$, there is, for every pointed F-CW-complex X, an isomorphism

$$Th_*^V(X)\colon \widetilde{\mathcal{H}}_*^F(X) \xrightarrow{\cong} \widetilde{\mathcal{H}}^F_{*+\dim_{\mathbb{R}}(V)}(V^\infty \wedge X),$$

called *equivariant suspension isomorphism* or *product Thom isomorphism*, which is natural in X; we also require that

$$Th_n^{V_1\oplus V_2}(X) = Th^{V_1}_{n+\dim_{\mathbb{R}}(V_2)}(V_2^\infty \wedge X) \circ Th_n^{V_2}(X),$$

for a pointed F-CW-complex X, $n \in \mathbb{Z}$, and $V_1, V_2 \in \operatorname{Rep}(F)$, where we identify the F-spaces $(V_1 \oplus V_2)^\infty \wedge X$ and $V_1^\infty \wedge (V_2^\infty \wedge X)$ as usual. Note that the name 'product Thom isomorphism' is justified, for a finite F-CW-complex X, by the F-homeomorphism $(V \times X)^\infty \cong V^\infty \wedge X_+$, so that $Th_*^V(X_+)$ is a map

$$Th_*^V(X_+)\colon \mathcal{H}_*^F(X) \xrightarrow{\cong} \widetilde{\mathcal{H}}^F_{*+\dim_{\mathbb{R}}(V)}\big((V\times X)^\infty\big),$$

and $V \times X$ is a product F-equivariant complex vector bundle over X. Suppose further that $\mathcal{H}_*^F(-)$ is multiplicative with unit $1_{\mathcal{H}} \in \mathcal{H}_0^F(pt)$. Define the *Euler class* $e^F_{\mathcal{H}}(V) \in \mathcal{H}^F_{-\dim_{\mathbb{R}}(V)}(pt)$ of $V \in \operatorname{Rep}(F)$ as the image of $1_{\mathcal{H}}$ under the composite

$$\mathcal{H}_0^F(pt) = \widetilde{\mathcal{H}}_0^F(S^0) \xrightarrow{\widetilde{\mathcal{H}}_0^F(\iota_V)} \widetilde{\mathcal{H}}_0^F(V^\infty) \xrightarrow{Th^V_{-d}(S^0)^{-1}} \widetilde{\mathcal{H}}^F_{-d}(S^0) = \mathcal{H}^F_{-d}(pt),$$

where $\iota_V\colon S^0 = pt_+ = \{0, \infty\} \hookrightarrow V^\infty$, and $d := \dim_{\mathbb{R}}(V)$.

We denote by 1_F the trivial one-dimensional complex representation of a finite group F. For convenience, we make the following definition, that we will need on various occasions in the sequel.

Definition 2.1. *Let $\{\mathcal{H}_*^F(-)\}_{F\in\mathscr{F}in\mathscr{G}rps}$ be a collection of stable, multiplicative F-equivariant homology theories with unit and with restriction structure. Let F be a finite group. We say that the Euler class $e^F_{\mathcal{H}}$ is* nimble *if*

(i) *it is* pointed *in the sense that $e^F_{\mathcal{H}}(1_F) = 0$;*
(ii) *it is* exponential, *i.e. $e^F_{\mathcal{H}}(V_1 \oplus V_2) = e^F_{\mathcal{H}}(V_1)\cdot e^F_{\mathcal{H}}(V_2)$, for $V_1, V_2 \in \operatorname{Rep}(F)$;*
(iii) *it is* compatible with restriction homomorphisms, *that is, for $V \in \operatorname{Rep}(F)$ and $H \leqslant F$, $e^H_{\mathcal{H}}(\operatorname{res}^F_H V) = \operatorname{res}^F_H(e^F_{\mathcal{H}}(V))$ holds.*

Example 2.2. *As we will see in Proposition 3.2 below, the Euler class is nimble for equivariant K-homology for finite groups.*

Finally, given a (non-trivial) group G, we denote by $\mathcal{P}r(G)$ the set of proper subgroups of G.

3. General form of Theorem 1.3

Using the terminology introduced in Section 2 for equivariant homology theories (in particular Definition 2.1), we can state the general form of Theorem 1.3.

Theorem 3.1. *Let $\mathcal{H}_*^?(-)$ be an equivariant homology theory. Suppose that $\mathcal{H}_*^?(-)$ has a restriction structure, and that for every finite group F and every finite non-cyclic group F', the following properties hold:*

(a) *$\mathcal{H}_*^F(-)$ is stable;*
(b) *$\mathcal{H}_*^F(-)$ is multiplicative with unit;*
(c) *the Euler class $e^F_{\mathcal{H}}$ is nimble;*

(d) *the following map is injective:*

$$\prod_{H\in\mathcal{P}r(F')} \mathrm{res}_H^{F'} : \mathcal{H}^{F'}_{2-2|F'|}(pt) \hookrightarrow \prod_{H\in\mathcal{P}r(F')} \mathcal{H}^{H}_{2-2|F'|}(pt)\,.$$

Let G be a group, and let $f\colon X\to Y$ be a G-map between proper G-CW-complexes. Consider the following properties:

(i) *the induced map $\mathcal{H}^C_*(f)\colon \mathcal{H}^C_*(X)\to\mathcal{H}^C_*(Y)$ is an isomorphism for every finite cyclic subgroup C of G;*
(ii) *the induced map $\mathcal{H}^F_*(f)\colon \mathcal{H}^F_*(X)\to\mathcal{H}^F_*(Y)$ is an isomorphism for every finite subgroup F of G;*
(iii) *the induced map $\mathcal{H}^G_*(f)\colon \mathcal{H}^G_*(X)\to\mathcal{H}^G_*(Y)$ is an isomorphism.*

Then, (i) and (ii) are equivalent, and they imply (iii).

The proof of this result will occupy Sections 4 and 5. In view of the following result, Theorem 1.3 is a consequence of Theorem 3.1.

Proposition 3.2. *Equivariant K-homology $K^?_*(-)$ is an equivariant homology theory with restriction structure. For any group G, $K^G_*(-)$ is a multiplicative G-equivariant homology theory with unit. For any finite group F, $K^F_*(-)$ is a stable F-equivariant homology theory and its Euler class is nimble; under Bott periodicity $K^F_*(-)\cong K^F_{*-2}(-)$ and via the ring isomorphism between $K^F_0(pt)$ and the complex representation ring $R(F)$ of F, the Euler class $e^F_K(V)$ of $V\in\mathrm{Rep}(F)$ corresponds to the usual Euler class $e(V)$ in representation theory:*

$$K^F_{-\dim_{\mathbb{R}}(V)}(pt)\ni e^F_K(V) \quad\longleftrightarrow\quad e(V):=\sum_{j=0}^{\dim_{\mathbb{C}}(V)}(-1)^j\,[\Lambda^jV]\in R(F)\,.$$

Furthermore, for a finite non-cyclic group F', the following map is injective:

$$\prod_{H\in\mathcal{P}r(F')} \mathrm{res}_H^{F'} : K^{F'}_*(pt) \hookrightarrow \prod_{H\in\mathcal{P}r(F')} K^{H}_*(pt)\,.$$

Before the proof, let us say a few words about equivariant K-homology. We start with the definition via the Davis-Lück approach [7]. Let G be a group, and Or(G) the orbit category of G, whose objects are orbits G/H, where $H\leqslant G$, and with G-maps as morphisms. Let Ω-SPECTRA be the category of Ω-spectra and consider

$$\underline{\mathbb{K}}^{\mathrm{top}}_G=\underline{\mathbb{K}}^{\mathrm{top}}_G(G/?)\colon \mathrm{Or(G)}\longrightarrow \Omega\text{-}\mathrm{SPECTRA}\,,\quad \mathrm{G/H}\longmapsto \underline{\mathbb{K}}^{\mathrm{top}}_{\mathrm{G}}(\mathrm{G/H})\,,$$

the Or(G)-Ω-spectrum (i.e. the covariant functor) constructed in [7] (see also [13]). Recall that its fundamental property, besides functoriality, is that

$$\pi_*(\underline{\mathbb{K}}^{\mathrm{top}}_G(G/H))\cong K_*(C^*_rH)\,,$$

canonically, for every $H\leqslant G$, where C^*_rH is the *reduced C^*-algebra* of H. For a G-CW-complex X, one considers the spectrum $X^?_+\otimes_{\mathrm{Or(G)}}\underline{\mathbb{K}}^{\mathrm{top}}_G(G/?)$ and defines

$$K^G_*(X):=\pi_*\big(X^?_+\otimes_{\mathrm{Or(G)}}\underline{\mathbb{K}}^{\mathrm{top}}_G(G/?)\big)\,,$$

where $X^?_+$ is the contravariant functor from Or(G) to the category of CW-complexes, taking G/H to X^H_+ (see [7] for the definition of the tensor product over the orbit category). For a G-CW-pair (X,A), we define $K^G_*(X,A)$ merely as

$\widetilde{K}_*^G(X/A)$. This theory satisfies Bott periodicity, that is, for a G-CW-pair (X, A), there is a canonical and natural isomorphism

$$\mathscr{B}_*^G(X,A)\colon K_*^G(X,A) \xrightarrow{\cong} K_{*+2}^G(X,A)\,.$$

Note that for $H \leqslant G$, $(G/H)_+^? \otimes_{\mathrm{Or(G)}} \mathbb{K}_G^{\mathrm{top}}(G/?)$ identifies with the spectrum $\mathbb{K}_G^{\mathrm{top}}(G/H)$, so that $K_*^G(G/H) \cong K_*(C_r^*H)$ in a canonical way. In particular, taking $H = G$, one has $G/G = pt$ and $K_*^G(pt) \cong K_*(C_r^*G)$. For *proper* G-CW-complexes, there is another approach, namely in terms of G-equivariant Kasparov KK-theory. Indeed, for a *proper* G-CW-complex X, there is a natural isomorphism

$$K_*^G(X) \cong \operatorname*{colim}_Y KK_*^G(C_0(Y),\, \mathbb{C})\,,$$

where the colimit is taken over the G-compact G-subspaces of X (or equivalently over the G-compact G-sub-CW-complexes of X). For details, we refer to [14] and [20]. The reader may be reassured by the fact that these explicit constructions and definitions will not be used here (except for the isomorphism $K_*^G(pt) \cong K_*(C_r^*G)$).

Proof of Proposition 3.2. The general statements for $K_*^?(-)$ and $K_*^G(-)$, and the fact that $K_*^F(-)$ is multiplicative with unit are folklore (compare with [16, pp. 201–202]). By [15], as in the diagram on page 301 of [10], $K_*^F(-)$ is stable and the Euler class $e_K^F(V)$ identifies with $e(V)$ as stated. Using this identification, to prove nimbleness of e_K^F, it suffices to note that $e(1_F) = [\mathbb{C}] - [\mathbb{C}] = 0$; that, given $V_1,\, V_2 \in \operatorname{Rep}(F)$, a direct computation based on the classical formula

$$\Lambda^j(V_1 \oplus V_2) \cong \bigoplus_{k+\ell=j} \Lambda^k V_1 \otimes \Lambda^\ell V_2\,,$$

implies that $e(V_1 \oplus V_2) = e(V_1)\cdot e(V_2)$; and that one has $e(\operatorname{res}_H^F V) = \operatorname{res}_H^F\big(e(V)\big)$ for $H \leqslant F$. Finally, by standard representation theory, for a finite group F, we have, for $\mathcal{FC}(F)$ the set of cyclic subgroups of F, an injection

$$\prod_{C\in\mathcal{FC}(F)} \operatorname{res}_C^F\colon R(F) \hookrightarrow \prod_{C\in\mathcal{FC}(F)} R(C)\,.$$

So, the injectivity statement follows from the inclusion $\mathcal{FC}(F') \subseteq \mathcal{P}r(F')$, and, for $n \in \mathbb{Z}$, from the equality $K_{2n+1}^{F'}(pt) = 0$, the isomorphism $K_{2n}^{F'}(pt) \cong R(F')$ and the obvious compatibility of the latter with the restriction homomorphisms. $\square$

4. Restriction to finite subgroups

The following general principle will permit us to reduce the proof of Theorem 3.1 to the case of finite groups.

Proposition 4.1. *Let $\mathcal{H}_*^?(-)$ be an equivariant homology theory and let $f\colon X \to Y$ be a G-map between G-CW-complexes. Assume that for every $H \leqslant G$ such that the fixed-point set $(X \amalg Y)^H$ is not empty, the induced map*

$$\mathcal{H}_*^H(f)\colon \mathcal{H}_*^H(X) \xrightarrow{\cong} \mathcal{H}_*^H(Y)$$

is an isomorphism, as indicated. Then, the following map is an isomorphism too:

$$\mathcal{H}_*^G(f)\colon \mathcal{H}_*^G(X) \xrightarrow{\cong} \mathcal{H}_*^G(Y)\,.$$

In the proof of Proposition 4.1 we will make use of the following three lemmas. The first one is obvious, but we state it as a lemma for later reference.

Lemma 4.2. *Let G be a group, H a subgroup of G, and Z a G-space. Consider $G/H\times Z$ as a G-space via the diagonal action: $\gamma\cdot(gH,\, z) := (\gamma gH, \gamma z)$ for $\gamma, g \in G$ and $z\in Z$. Then, the following map is a canonical G-homeomorphism:*

$$h_H^G(Z)\colon G\times_H Z \xrightarrow{\cong} G/H\times Z\,,\quad [g,z]\longmapsto (gH, gz)\,.$$

It is natural both in Z and H. The inverse is given by $(gH,z)\mapsto [g,g^{-1}z]$. $\square$

The second lemma is standard and follows straightforwardly from the defining properties of G-equivariant homology theories. Before we state it, recall that by a *family* of subgroups $\mathcal{F}=\mathcal{F}(G)$ of a group G, we mean a *non-empty* set of subgroups which is closed under conjugation and passing to subgroups; a G-CW-complex is called *$\mathcal{F}$-free*, if all of its isotropy groups lie in $\mathcal{F}$.

Lemma 4.3. *Let $\mathcal{F}$ be a family of subgroups of a group G. Consider a natural transformation $\tau_*(-)\colon \mathcal{H}_*^G(-)\to\mathcal{K}_*^G(-)$ of G-equivariant homology theories. If $\tau_*(G/H)$ is an isomorphism for every $H\in\mathcal{F}$, then $\tau_*(Z)$ is an isomorphism for every $\mathcal{F}$-free G-CW-complex Z.* $\square$

If $\mathcal{F}$ is a family of subgroups of G, we write $E(G,\mathcal{F})$ for its classifying space. We recall that $E(G,\mathcal{F})$ is characterized by the fact that it is $\mathcal{F}$-free and that for any $\mathcal{F}$-free G-CW-complex X there is, up to G-homotopy, a unique G-map $X\to E(G,\mathcal{F})$.

Lemma 4.4. *For a family $\mathcal{F}$ of subgroups of a group G and an $\mathcal{F}$-free G-CW-complex Z, the projection map $pr_Z\colon E(G,\mathcal{F})\times Z\to Z$ is a G-homotopy equivalence with, as G-homotopy inverse, the unique G-homotopy class $in_Z\colon Z\to E(G,\mathcal{F})\times Z$, given by the universal property of $E(G,\mathcal{F})$ and with id_Z as second component.* $\square$

Proof of Proposition 4.1. The map f induces a natural transformation

$$\mathcal{H}_*^G(-\times f)\colon \mathcal{H}_*^G(-\times X)\longrightarrow \mathcal{H}_*^G(-\times Y)$$

of G-equivariant homology theories (the G-action on products is the diagonal one). Let $\mathcal{F}$ denote the family of all subgroups of G for which $(X\amalg Y)^H$ is non-empty. Given $H\in\mathcal{F}$, the induced map

$$\mathcal{H}_*^G(G/H\times f)\colon \mathcal{H}_*^G(G/H\times X)\longrightarrow \mathcal{H}_*^G(G/H\times Y)$$

is an isomorphism, because, by Lemma 4.2 and by making use of the restriction structure, $\mathcal{H}_*^G(G/H\times -)$ is naturally isomorphic to $\mathcal{H}_*^H(-)$, and, by assumption, $\mathcal{H}_*^H(f)\colon \mathcal{H}_*^H(X)\to\mathcal{H}_*^H(Y)$ is an isomorphism. It follows then from Lemma 4.3, by choosing Z to be the $\mathcal{F}$-free G-CW-complex $E(G,\mathcal{F})$, that the induced map

$$\mathcal{H}_*^G\big(E(G,\mathcal{F})\times f\big)\colon \mathcal{H}_*^G\big(E(G,\mathcal{F})\times X\big)\longrightarrow \mathcal{H}_*^G\big(E(G,\mathcal{F})\times Y\big)$$

is an isomorphism too. Since X and Y are $\mathcal{F}$-free (by very choice of $\mathcal{F}$), applying Lemma 4.4, we get a composition of G-maps

$$X \xrightarrow[in_X]{\overset{G}{\simeq}} E(G,\mathcal{F})\times X \xrightarrow{\mathrm{id}\times f} E(G,\mathcal{F})\times Y \xrightarrow[pr_Y]{\overset{G}{\simeq}} Y\,,$$

which is both G-homotopic to f and an $\mathcal{H}_*^G$-isomorphism, finishing the proof. $\square$

Remark 4.5. *Proposition 4.1 proves that (ii) implies (iii) in Theorem 3.1 (note that the properness assumption on X and Y is used to assure that $(X\amalg Y)^H$ is empty for H an infinite group). It is clear that (i) follows from (ii) in Theorem 3.1. Thus, it remains to prove that (i) implies (ii) to establish Theorems 1.3 and 3.1; this is precisely what Proposition 5.1 below does.*

5. Restriction from finite subgroups to finite cyclic subgroups

In this section, we start by proving the following proposition; we conclude it by assembling the pieces to prove Theorems 1.3 and 3.1.

Proposition 5.1. *Let $\mathcal{H}_*^?$ be an equivariant homology theory for finite groups, with restriction structure. Suppose that properties (a)–(d) of Theorem 3.1 hold for every finite group F and every finite non-cyclic group F'. Let F be a finite group, and $f\colon X \to Y$ an F-map between F-CW-complexes. Consider the following statements:*

(i) *the induced map $\mathcal{H}_*^C(f)\colon \mathcal{H}_*^C(X) \to \mathcal{H}_*^C(Y)$ is an isomorphism for every finite cyclic subgroup C of F;*
(ii) *the induced map $\mathcal{H}_*^F(f)\colon \mathcal{H}_*^F(X) \to \mathcal{H}_*^F(Y)$ is an isomorphism.*

Then, (i) implies (ii).

We point out that the corresponding statement for G-equivariant K-theory $K_G^*(-)$ (i.e. G-equivariant K-*co*homology) and for X and Y finite G-CW-complexes with G a compact Lie group, is true too. Indeed, in [19], McClure proved that if $x \in K_G^*(X)$ restricts to zero in $K_F^*(X)$ for every finite subgroup F of G, then $x = 0$. Combined with [12], this gives the required result. Closely related ideas are contained in [6] and in the article [5], based on [4].

Proof of Proposition 5.1. We will proceed by induction on the order of F. For the induction step, it suffices to show that the forthcoming result holds, and then the proof is complete. □

Proposition 5.2. *Under the assumptions of 5.1, let F' be a finite non-cyclic group. If $\mathcal{H}_*^P(X) \to \mathcal{H}_*^P(Y)$ is an isomorphism for all proper subgroups $P < F'$, then $\mathcal{H}_*^{F'}(X) \to \mathcal{H}_*^{F'}(Y)$ is an isomorphism too.*

Proof. If $\mathcal{H}_*^P(X) \to \mathcal{H}_*^P(Y)$ is an isomorphism for all proper subgroups $P < F'$, then by applying 4.1, we see that $\mathcal{H}_*^{F'}\big(X \times E(F', \mathcal{P}r)\big) \to \mathcal{H}_*^{F'}\big(Y \times E(F', \mathcal{P}r)\big)$ is an isomorphism too, where $\mathcal{P}r$ stands for the family of proper subgroups of F'. To conclude the proof, it suffices therefore to show that the F'-homology theory $\mathcal{H}_*^{F'}\big(-\times\big(E(F', \mathcal{A}\ell\ell), E(F', \mathcal{P}r)\big)\big)$ is the zero theory. Let V_{red} denote the reduced regular representation of F' over $\mathbb{C}$. Clearly, every proper subgroup $H < F'$ has a fixed non-zero vector in V_{red} and, by nimbleness of $e_{\mathcal{H}}^H$, we infer that $e_{\mathcal{H}}^H(V_{\mathrm{red}}) = 0$ for all such H. By assumption (d) of 3.1 and by nimbleness of $e_{\mathcal{H}}^{F'}$, we therefore have $e_{\mathcal{H}}^{F'}(V_{\mathrm{red}}) = 0$. By [22, Satz 5], in our situation, there is a canonical isomorphism of F'-equivariant homology theories

$$\{e_{\mathcal{H}}^{F'}(V_{\mathrm{red}})\}^{-1}\mathcal{H}_*^{F'}(-) \cong \mathcal{H}_*^{F'}\Big(-\times\big(E(F', \mathcal{A}\ell\ell), E(F', \mathcal{P}r)\big)\Big),$$

where the left-hand side denotes the localization with respect to the multiplicative subset of $\mathcal{H}_*^{F'}(pt)$ generated by $e_{\mathcal{H}}^{F'}(V_{\mathrm{red}})$; so, $\{e_{\mathcal{H}}^{F'}(V_{\mathrm{red}})\}^{-1}\mathcal{H}_*^{F'}(-)$ is the zero theory, simply because $e_{\mathcal{H}}^{F'}(V_{\mathrm{red}}) = 0$. □

Remark 5.3. *Let F be a finite group and $\mathcal{F}$ a family of subgroups of F. It follows from Propositions 3.2 and 5.1 that, provided $\mathcal{F}$ contains $\mathcal{FC}$, the constant map from $E(F, \mathcal{F})$ induces, for $n \in \mathbb{Z}$, the following isomorphism (compare with* [10]*):*

$$K_n^F\big(E(F, \mathcal{F})\big) \xrightarrow{\cong} K_n^F(pt) \cong \begin{cases} R(F), & \text{if } n \text{ is even} \\ 0, & \text{otherwise}. \end{cases}$$

Proof of Theorem 3.1. As observed in Remark 4.5, the result is a consequence of Propositions 4.1 and 5.1. □

Proof of Theorem 1.3. This follows from Theorem 3.1 and Proposition 3.2. □

6. Applications to the Baum-Connes conjecture

In this section, we give some applications of our results in the framework of the Baum-Connes conjecture.

The Baum-Connes conjecture states that for a countable discrete group G, the *Baum-Connes assembly map* (or *analytic assembly map* or *G-index map*)

$$\mu_*^G : K_*^G\big(E(G,\mathcal{F}in)\big) \longrightarrow K_*^G(pt) \cong K_*(C_r^* G)\,,$$

induced by the constant map on $E(G,\mathcal{F}in)$, is an isomorphism [1, 2, 3, 20]. (To be precise, this is the reformulation à la Davis-Lück [7] of the conjecture, which, by work of Hambleton-Pedersen [11], is equivalent to the usual form.) In fact, μ_*^G is defined for arbitrary discrete groups (i.e. not necessarily countable) and the statement of the conjecture makes sense in this generality. The slogan is that one would like to understand (or compute) the group $K_*(C_r^*G)$ of analytic nature, and that $K_*^G\big(E(G,\mathcal{F}in)\big)$ is of geometric and topological nature, and is in principle computable, compare [1, 16, 18, 20]. In the sequel, we identify $K_*^G(pt)$ with $K_*(C_r^*G)$.

Theorem 6.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of subgroups of a group G, satisfying $\mathcal{FC}(G) \subseteq \mathcal{F} \subseteq \mathcal{F}in(G)$ and $\mathcal{F} \subseteq \mathcal{G}$. Suppose that every group in $\mathcal{G}$ satisfies the Baum-Connes conjecture. Then, for an arbitrary $\mathcal{G}$-free G-CW-complex Z, the G-map $\rho_G^{\mathcal{F},\mathcal{G}} : E(G,\mathcal{F}) \to E(G,\mathcal{G})$ (unique up to G-homotopy) induces the following isomorphism in G-equivariant K-homology, that is natural in Z:*

$$K_*^G(\mathrm{id}_Z \times \rho_G^{\mathcal{F},\mathcal{G}}) : K_*^G\big(Z \times E(G,\mathcal{F})\big) \xrightarrow{\cong} K_*^G\big(Z \times E(G,\mathcal{G})\big)\,.$$

Moreover, G satisfies the Baum-Connes conjecture if and only if the map

$$K_*^G\big(E(G,\mathcal{G})\big) \longrightarrow K_*(C_r^*G)\,,$$

induced by the projection onto the point, is an isomorphism.

For closely related results, we refer to [9, Thm. A.10], to [21, Thm. 2.6.1], to [17, Thm. 2.3] and to [8, Cor. 3.6].

Proof. Observe that for every $H \in \mathcal{G}$, the map

$$K_*^H(\rho_G^{\mathcal{F}in,\mathcal{G}}) : K_*^H\big(E(G,\mathcal{F}in)\big) \longrightarrow K_*^H\big(E(G,\mathcal{G})\big)$$

is an isomorphism. Indeed, both sides are naturally isomorphic to $K_*(C_r^*H)$, because $K_*^H\big(E(G,\mathcal{F}in)\big) \cong K_*^H\big(E(H,\mathcal{F}in)\big) \cong K_*(C_r^*H)$ since $E(G,\mathcal{F}in)$ is H-homotopy equivalent to $E(H,\mathcal{F}in)$ and H satisfies the Baum-Connes conjecture, and because $K_*^H\big(E(G,\mathcal{G})\big) \cong K_*(C_r^*H)$ as $E(G,\mathcal{G})$ is H-contractible. By Theorem 1.3, since $E(G,\mathcal{F})$ and $E(G,\mathcal{F}in)$ are C-contractible for $C \leqslant G$ finite cyclic, the natural transformations of G-homology theories

$$K_*^G\big(-\times E(G,\mathcal{F})\big) \longrightarrow K_*^G\big(-\times E(G,\mathcal{F}in)\big) \longrightarrow K_*^G\big(-\times E(G,\mathcal{G})\big)$$

are therefore isomorphisms on any orbit G/H with $H \in \mathcal{G}$, hence the first assertion. The second part of the theorem follows straightforwardly from the first and from Theorem 1.3, since for $C \leqslant G$ finite cyclic, $E(G,\mathcal{F}in)$ and $E(G,\mathcal{G})$ are C-contractible, so that $K_*^G\big(E(G,\mathcal{F}in)\big) \cong K_*^G\big(E(G,\mathcal{G})\big)$. □

The next result illustrates 6.1, using the fact that amenable groups satisfy the Baum-Connes conjecture.

Corollary 6.2. *Let G be an arbitrary group. Then one has natural isomorphisms*

$$K_*^G\big(E(G,\mathcal{FC})\big) \cong K_*^G\big(E(G,\mathcal{F}in)\big) \cong K_*^G\big(E(G,\mathcal{VC})\big) \cong K_*^G\big(E(G,\mathscr{A}m)\big)\,,$$

induced by the inclusions $\mathcal{FC} \subset \mathcal{F}in \subset \mathcal{VC} \subset \mathscr{A}m$ where $\mathcal{VC}$ (respectively $\mathscr{A}m$) stands for the family of virtually cyclic (respectively amenable) subgroups of G. □

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University of Lausanne, IGAT, BCH, EPFL, CH-1015 Lausanne, Switzerland
E-mail address: michel.matthey@ima.unil.ch

Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland
E-mail address: mislin@math.ethz.ch