COHOMOLOGICALLY CENTRAL ELEMENTS AND FUSION IN GROUPS

Guido Mislin

Mathematik and Department of Mathematics
ETH-Zentrum Ohio State University
8092 – Zürich Columbus, Ohio 43210

Introduction.

Let G be a compact Lie group with center ZG. For p a prime we put

$$_pZG = \{x \in ZG | x^p = 1\},$$

the maximal elementary abelian p-subgroup of ZG. The cohomology algebra $H^*(BG; \mathbb{F}_p)$ is an unstable algebra over A_p, the mod-p Steenrod algebra. For an arbitrary unstable A_p-algebra R the Dwyer-Wilkerson center ZR of R is defined by

$$ZR := \{f : R \to H^*(B\mathbb{Z}/p; \mathbb{F}_p) \mid T_f : R \to T_f R \text{ an isomorphism}\}.$$

Here, T_f denotes the relative version of Lannes' T-functor, which is defined by $T_f R = TR \otimes_{T^\circ R} \mathbb{F}_p$, where $\mathbb{F}_p$ is considered as a module over the degree zero component $T^\circ R$ of TR via the adjoint $TR \to \mathbb{F}_p$ of $f : R \to H^*(B\mathbb{Z}/p; \mathbb{F}_p)$. It is shown in [DW] that ZR has a natural abelian group structure. There is an obvious group homomorphism

$$\tilde{\varphi} : {_pZG} \to ZH^*(BG; \mathbb{F}_p)$$

which arises as follows. Let x be in $_pZG$, with associated map $\phi(x) : \mathbb{Z}/p \to G$ and $f = (B\phi(x))^* : H^*(BG; \mathbb{F}_p) \to H^*(B\mathbb{Z}/p; \mathbb{F}_p)$. Such an f lies in $ZH^*(BG; \mathbb{F}_p)$, because T_f corresponds to the map induced by the inclusion of the centralizer $C_G(x)$ in G, which is an isomorphism; one then defines $\tilde{\varphi}(x) := f$. It is clear that $\tilde{\varphi}$ is injective, because from basic properties of the T-functor [L] one knows that for any

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compact Lie group, $\tilde{\varphi}(x) = \tilde{\varphi}(y)$ implies that x and y are conjugate in G, thus equal, since they are central. However, in general φ will not be surjective (for instance the inclusion of an element of order two in the symmetric group S_3 corresponds to a non trivial element in $ZH^*(BS_3; \mathbb{F}_2)$, but the center of S_3 is trivial). Note also that ${}_pZG$ injects into ${}_pZ(G/H)$, if $H < G$ is a normal p'-group (a torsion group all of whose elements are of order prime to p) and, if G is finite, one has $H^*(BG; \mathbb{F}_p) = H^*(BG/H; \mathbb{F}_p)$ for such an H. It is therefore natural to consider the quotient $G/O_{p'}(G)$, where $O_{p'}(G)$ denotes the largest normal p'-subgroup of G. Note that $O_{p'}(G)$ may be an infinite group and it is totally disconnected in the Lie group G. Of course, one can still define ${}_pZ(G/O_{p'}(G))$ as before, although $G/O_{p'}(G)$ may fail to be a Lie group (e.g. consider the case of $G = S^1$). We will see that, however, one has for an arbitrary compact Lie group G a well defined map ${}_pZ(G/O_{p'}(G)) \to H^*(BG; \mathbb{F}_p)$, and we denote this map by φ. Our main theorem can then be expressed as follows.

Theorem 1. *Let G be a compact (not necessarily connected) Lie group and p a fixed prime. Then the canonical map*

$$\varphi : {}_pZ(G/O_{p'}(G)) \to ZH^*(BG; \mathbb{F}_p)$$

is an isomorphism of abelian groups.

As we will see in the course of the proof of this theorem, for an element x in ${}_pZ(G/O_p'(G))$ any counter image $\tilde{x}$ in G will have the property that the inclusion $C_G(\tilde{x}_p) \to G$ is a mod-p homology isomorphism on the classifying space level, where $\tilde{x}_p$ denotes the p-part of the torsion element $\tilde{x}$. Obviously, $\tilde{x}_p$ satisfies $(\tilde{x}_p)^p = 1$, and we will show that $\tilde{x}_p$ depends up to conjugation in G only on its projection x in $G/O_{p'}(G)$; it seems natural to call $\tilde{x}_p$ a "p-cohomologically central element of G". Conversely, an element y in G satisfying $y^p = 1$ and such that $C_G(y) \to G$ induces a mod-p cohomology isomorphism on the classifying space level, will necessarily be of the form $\tilde{x}_p$ for some x in ${}_pZ(G/O_p'(G)$, as we will see. Our Theorem 1 can therefore be rephrased as follows.

Theorem 2. *Let G be a compact Lie group and p a prime. Then there is a natural bijection between the set of conjugacy classes of p-cohomologically central elements y in G satisfying $y^p = 1$, and the group ${}_pZ(G/O'_p(G))$.*

It seems natural to widen the scope a little bit and to consider subsets rather than elements of G, which are cohomologically central in the following sense.

Definition. Let G be a (topological) group and S an arbitrary subset of G. We call S p-cohomologically central in G, if the inclusion of (topological) groups $C_G(S) \to G$ induces an isomorphism in mod-p cohomology on the level of classifying spaces.

We will mainly be interested in the case where G is a compact Lie group. In that case, $C_G(S)$ will be a closed subgroup of G, thus a compact Lie group too; note also that $C_G(S) = C_G(S(\alpha))$ for a suitable finite subset $S(\alpha)$ of S. This can be seen as follows. Obviously, $C_G(S)$ is the intersection of groups $C_G(S(\beta))$ where $S(\beta)$ runs over the finite subsets of S, and each $C_G(S(\beta))$ is compact. But the compact subgroups in a compact Lie group satisfy the descending chain condition, and the result follows easily.

In Section 1, we will discuss fusion from a cohomological point of view and prove Theorems 1 and 2. In Section 2, we discuss two applications, one dealing with finite p-groups of exponent p, and one concerning the cohomology of finite solvable groups.

1) Fusion.

We recall first the classical notion of fusion in the setting of finite groups. Let G be a finite group, p a prime and $P < G$ a Sylow p-subgroup. One says that a subgroup $H < G$ containing P *controls the p-fusion* in G if for any subgroup π of P and any element g in G such that $\pi^g = g^{-1}\pi g < P$ one has $g = ch$ for some c in $C_G(\pi)$ and h in H. For instance, if P is abelian then by a classical theorem of Burnside, the normalizer $N_G(P)$ controls p-fusion in G. Note also that if $H < G$ controls the p-fusion in G, then G and H have the same mod-p cohomology as one deduces immediately from the description of the mod-p cohomology by means of stable elements in the cohomology of P.

To generalize the notion of fusion to arbitrary groups, we proceed as follows. For G any group and p a prime, the Frobenius category $\mathrm{Frob}_p(G)$ is defined as category with objects the finite p-subgroups of G, and morphisms the group homomorphisms of the form $\mathrm{in}(g) : x \mapsto g^{-1}xg$, for some g in G.

Definition ([MT]). Let G be an arbitrary group, H a subgroup, and p a prime. Then H *controls finite p-fusion* in G, if the inclusion $H \to G$ induces an equivalence of Frobenius categories $\mathrm{Frob}_p(H) \to \mathrm{Frob}_p(G)$.

Fusion is tightly linked to cohomology by the following theorem.

Fusion-Theorem ([M]). *Let $f : H \to G$ be a morphism of compact Lie groups, and p a prime. Then the following are equivalent:*

(i) $(Bf)^* : H^*(BG; \mathbb{F}_p) \to H^*(BH; \mathbb{F}_p)$ *is an isomorphism*

(ii) f *induces an equivalence of categories* $Frob_p(H) \to Frob_p(G)$.

The basic ingredient of the proof of Theorem 1 is the following theorem, which links fusion with the internal structure of a group.

Z*-Theorem ([MT]). *Let G be a compact Lie group and p a prime. Let $A < G$ be a p-subgroup (not necessarily finite), or a p-toral subgroup. If $C_G(A)$ controls finite p-fusion in G, then*

$$G = C_G(A)O_{p'}(G) = C_G(A)[A, G]$$

and the commutator group $[A, G]$ is a (normal) finite p'-subgroup of G.

Remark. The reader who is primarily interested in the case of finite groups, should consult [B] for a reduction of the Z^*-theorem for finite groups to the case of finite simple groups. The Z^*-theorem for finite simple groups can then be checked case by case. Unfortunately, no proof of the Z^*-theorem not using the classification is known for finite groups and p an odd prime. However, for the applications offered in Section 2 , one only needs the Z^*-theorem for finite solvable groups. In that case, the methode of reduction as described in [B] provides a complete proof, since for finite solvable simple groups (i.e. finite cyclic groups of prime order) the statement is trivial. In [MT], the Z^*-theorem for compact Lie groups is proved by reducing it to the case of finite groups; no new proof is offered in the case of finite groups.

Proof of Theorem 1. We define $\varphi : {}_pZ(G/O_{p'}(G)) \to ZH^*(BG;\mathbb{F}_p)$ as indicated above. For $x \in {}_pZ(G/O_{p'}(G))$ we choose $y \in G$ satisfying $y^p = 1$, and y over x. Then $[y, G] < O_{p'}(G)$, and $[y, G]$ is finite, since $[y, G^0] = \{1\}$ (it is connected and totally disconnected). Thus $G \to G/[y, G]$ is a morphism of compact Lie groups, which induces a mod-p cohomology isomorphism on the level of classifying spaces. Since $\overline{y} \in G/[y, G]$ is central, it follows by the naturality of Lannes' T-functor that $y \in G$ is p-cohomologically central in G (that is, $C_G(\{y\}) \to G$ is a mod-p cohomology isomorphism); we then put

$$\varphi(x) = f(y) : H^*(BG;\mathbb{F}_p) \to H^*(B\mathbb{Z}/p;\mathbb{F}_p),$$

where $f(y)$ is the map induced by the map $\mathbb{Z}/p \to \langle y\rangle \to G$, ($y$ corresponding to the residue class of 1 in $\mathbb{Z}/p$). To see that $\varphi(x)$ is well defined, let z denote another element in G over x satisfying $z^p = 1$. Then z will also be p-cohomologically central in G and both, y and z, map to central elements in $G/([y, G]\cdot[z, G])$. We claim that y and z are conjugate in G, and therefore $f(y) = f(z)$. Namely, by construction, y and z are both in the torsion group $\langle y, O_{p'}(G)\rangle < G$ and, from the short exact sequence

$$O_{p'}(G) \to \langle y, O_{p'}(G)\rangle \to \langle x\rangle$$

we see that $\langle y\rangle$ and $\langle z\rangle$ are Sylow p-subgroups of $\langle y, O_{p'}(G)\rangle$. Since in a linear torsion group any two Sylow p-subgroups are conjugate [W], we conclude that $z^n = g^{-1}yg$ for some $g \in G$ and some n prime to p. Clearly, we can choose $n = 1$, because y and z both project onto x, and x, y, z all have the same order by construction. To see that φ is injective, suppose that $\varphi(x_1) = \varphi(x_2)$, and y_1, y_2 elements in G over x_1, x_2 satisfying $y_i^p = 1$, so that $f(y_1) = f(y_2) : H^*(BG;\mathbb{F}_p) \xrightarrow{i} H^*(\mathbb{Z}/p;\mathbb{F}_p)$. But then y_1 and y_2 are conjugate in G by one of the basic properties of Lannes' T-functor [L]. In particular, x_1 and x_2 will be conjugate, thus equal, because they are central in $G/O_{p'}(G)$; thus φ is injective. Now choose $f \in ZH^*(BG;\mathbb{F}_p)$, represented by $B\mathbb{Z}/p \to BG$ which, as is well known, is induced by a homomorphism $\rho : \mathbb{Z}/p \to G$; we used here the basic fact that for any compact Lie group G, one has natural bijections

$$mor(H^*(BG;\mathbb{F}_p), H^*(\mathbb{Z}/p;\mathbb{F}_p)) \cong [B\mathbb{Z}/p, BG] \cong Rep(\mathbb{Z}/p, G)$$

where $mor(\ , \ )$ stands for the set of A_p-algebra morphisms, and $Rep(\mathbb{Z}/p, G)$ for the set of conjugacy classes of homomorphism $\mathbb{Z}/p \to G$. Put now $y = \rho(1 \text{ mod-}p)$. Then y will be p-cohomologically central in G since f was chosen in $ZH^*(BG; \mathbb{F}_p)$. By the Z^*-theorem (loc. cit.) we thus have

$$G = C_G(y) O_{p'}(G),$$

which implies that y maps to an element $x \in {}_pZ(G/O_{p'}(G))$. By construction, $\varphi(x) = f$, showing that p is surjective and completing the proof of Theorem 1.

Note that our proof of Theorem 1 also shows that the conjugacy classes of elements $y \in G$ satisfying $y^p = 1$ and for which $C_G(y) \to G$ is a mod-p cohomology isomorphism, correspond bijectively to the elements of ${}_pZ(G/O_{p'}(G))$, as claimed in Theorem 2.

2) Two applications

For G a finite p-group, Theorem 1 implies that

$$ {}_pZ(G) \cong ZH^*(BG; \mathbb{F}_p). \tag{1}$$

Since $H^*(B\mathbb{Z}/p^2; \mathbb{F}_p) \cong H^*(B\mathbb{Z}/p^3; \mathbb{F}_p)$ as A_p-algebras, we cannot expect to be able to compute the order of the finite p-group G just from the A_p-algebra structure of $H^*(BG; \mathbb{F}_p)$. But if we assume G to be of exponent p, this is in principle possible, as one can see from the proof of the following proposition.

Proposition 1. *Let G and H be finite groups of exponent p, p a prime. Suppose that $H^*(BG; \mathbb{F}_p)$ and $H^*(BH; \mathbb{F}_p)$ are isomorphic as A_p-algebras. Then G and H have the same order.*

Proof. The assumption implies that the centers $Z(G)$ and $Z(H)$ are isomorphic since they are isomorphic to $ZH^*(BG; \mathbb{F}_p)$ and $ZH^*(BH; \mathbb{F}_p)$ respectively. Choose an A_p-algebra isomorphism

$$\lambda : H^*(BG; \mathbb{F}_p) \to H^*(BH; \mathbb{F}_p).$$

Then λ maps the set of A_p-algebra homomorphisms $mor(H^*(BH;\mathbb{F}_p), H^*(B\mathbb{Z}/p;\mathbb{F}_p))$ bijectively onto the corresponding set $mor(H^*(BG;\mathbb{F}_p), H^*(B\mathbb{Z}/p;\mathbb{F}_p))$. These sets correspond naturally to the sets of conjugacy classes of elements in H (respectively G), since $mor(H^*(BG;\mathbb{F}_p), H^*(B\mathbb{Z}/p;\mathbb{F}_p)) \cong Rep(\mathbb{Z}/p, G)$ and $Rep(\mathbb{Z}/p, G) \cong \{$ conjugacy classes in $G\}$, as G has exponent p (similarly for H). If we put

$$J(G) = mor(H^*(BG;\mathbb{F}_p), H^*(B\mathbb{Z}/p;\mathbb{F}_p)) \backslash ZH^*(BG;\mathbb{F}_p)$$

then λ induces a bijection $\lambda^* : J(H) \cong J(G)$. Of course, $J(G)$ (respectively $J(H)$) is in natural bijection with the set of conjugacy classes of non-central elements in G (respectively H). To compute the order $|G|$ of G, we consider the class equation

$$|G| = |Z(G)| + \sum_{J(G)} |G|/|C_G(x)|, \tag{2}$$

If the conjugacy class of $x \in G$ corresponds to $f(x) \in J(G)$, so that $f(x) : H^*(BG;\mathbb{F}_p) \to H^*(B\mathbb{Z}/p;\mathbb{F}_p)$, and $\lambda^*(f(y)) = f(x) \in J(G)$, $y \in H$, then by naturality of Lannes' T-functor one has

$$T_{f(x)}H^*(BG;\mathbb{F}_p) \cong T_{f(y)}H^*(BH;\mathbb{F}_p).$$

But $T_{f(x)}H^*(BG;\mathbb{F}_p) \cong H^*(BC_G(x);\mathbb{F}_p)$, and $T_{f(y)}H^*(BH;\mathbb{F}_p) \cong H^*(BC_H(x);\mathbb{F}_p)$, so that λ induces an A_p-algebra isomorphism

$$H^*(BC_G(x);\mathbb{F}_p) \cong H^*(BC_H(y);\mathbb{F}_p).$$

Since x (respectively y) is non-central, $C_G(x)$ is smaller then G (respectively $C_H(y)$ is smaller than H). By induction on the order of the groups involved, we infer $|C_G(x)| = |C_H(y)|$. Comparing now (2) with the corresponding formula

$$|H| = |Z(H)| + \sum_{J(H)} |H|/|C_H(y)|, \tag{3}$$

we see that $|G|$ and $|H|$ satisfy the same recursion formulas, which implies that $|G| = |H|$.

The second application involves the Fusion-Theorem and was suggested to me by Jacques Thévenaz.

Proposition 2. *Let* $f : H \to G$ *be a homomorphism of finite groups and assume that* G *is solvable. If for some prime* p *the induced map*

$$(Bf)^* : H^*(BG; \mathbb{F}_p) \to H^*(BH; \mathbb{F}_p)$$

is an isomorphism, then f *maps* $O_{p'}(H)$ *into* $O_{p'}(G)$ *and induces an isomorphism*

$$H/O_{p'}(H) \cong G/O_{p'}(G).$$

Proof. Clearly, $G \to G/O_{p'}(G)$ induces an isomorphism in mod-p cohomology and therefore our assumption on f implies that

$$\overline{f} : H \to G/O_{p'}(G) = \overline{G}$$

is a mod-p cohomology isomorphism too. Thus $ker(\overline{f})$ is a p'-group. If $\overline{f}$ is surjective, we conclude that $ker(\overline{f}) = O_{p'}(H)$ and we are done. Put $\overline{H} = H/ker(\overline{f})$. Then the inclusion $\overline{H} < \overline{G}$ will induce a mod-p cohomology isomorphism too and, by the Fusion-Theorem (loc. cit.), we conclude that $\overline{H}$ controls the p-fusion in $\overline{G}$. Now choose $\pi = O_p(\overline{G})$, the largest normal p-subgroup of $\overline{G}$. Since $\overline{G}$ is solvable,

$$C_{\overline{G}}(\pi) < O_p(\overline{G}) = \pi$$

(see [G, 8.1.1]) and, because $\overline{H}$ controls finite p-fusion, π is conjugate to a subgroup of $\overline{H}$, thus $\pi < \overline{H}$ since π is normal in $\overline{G}$. From the definition of the control of p-fusion we infer

$$N_{\overline{G}}(\pi) = N_{\overline{H}}(\pi) \cdot C_{\overline{G}}(\pi),$$

which implies $\overline{G} = \overline{H} \cdot C_{\overline{G}}(\pi)$ because π is normal in $\overline{G}$. But we observed already that $C_{\overline{G}}(\pi) < \pi$ and $\pi < \overline{H}$, thus $\overline{H} = \overline{G}$, completing the proof.

Remark. We only used that the finite group G satisfies

$$C_{\overline{G}}(O_p(\overline{G})) < O_p(\overline{G})$$

where $\overline{G} = G/O_{p'}(G)$; such a group G is called *p-constrained* (cf. [G]). For instance, p-solvable groups are p-constrained (G is called p-solvable, if G admits a subnormal series with quotients either p-groups or p'-groups [G]).

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