ON THE p-PRIMARY COHOMOLOGY OF OUT($\mathbf{F}_n$) IN THE p-RANK ONE CASE

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Abstract. Let p be an odd prime. We compute the p-primary cohomology of $Out(F_n)$ above its virtual cohomological dimension, for the first three cases for which the p-rank of $Out(F_n)$ equals one, namely $n = p-1, p$ and $p+1$.

Introduction

Let F_n be a free group of rank n and $Out(F_n) = Aut(F_n)/Inn(F_n)$ its group of outer automorphisms. It is known that $Out(F_n)$ has finite virtual cohomological dimension (vcd), equal to 0 if $n = 1$, and $2n-3$ if $n > 1$, see [CV]. In the sequel p will always denote a prime number. We say that the group G has p-rank one if $\mathbb{Z}/p \subset G$ but $\mathbb{Z}/p \times \mathbb{Z}/p \not\subset G$. According to [GMV], $Out(F_n)$ has p-rank one if and only if $p-1 \leq n < 2(p-1)$. In this note we compute the p-primary part of the cohomology of $Out(F_n)$ for $n = p-1$, p and $p+1$ in the range above the vcd of $Out(F_n)$. For $n = p-1$ we recover the result proved in [GMV]; but the method used here yields, in addition, some information concerning the cohomology in the range below the vcd. The cases $n = p$ and $n = p+1$ are more complex and new. One has to assume $p \geq 5$ for $n = p+1$, to fit n within the range $p-1 \leq n < 2(p-1)$. It is well-known that the p-primary cohomology of a p-rank one group of finite vcd is periodic above the vcd, and depends there only upon the p-primary cohomology of the normalizers of the subgroups of order p. It turns out that, rather than studying these normalizers directly, it is much more effective to compute the cohomology in question by letting $Out(F_n)$ act on the simplicial spine X_n of "outer space" introduced by Culler-Vogtmann [CV] (see also [SV]). The space X_n is a $2n-3$-dimensional contractible simplicial complex, on which $Out(F_n)$ acts with finite stabilizers. Our results stated below then follow by studying this action. It is convenient to formulate our results using coefficients in $\mathbb{Z}_{(p)}$, the integers localized at $(p) \subset \mathbb{Z}$. The well-known cohomology of the following basic finite groups will enter: the symmetric group Σ_p of order $p!$, the dihedral group D_{2p} of order $2p$ (here p denotes an odd prime), and M_{4p}, the metacyclic group of order $4p$

$$M_{4p} = \mathbb{Z}/p \rtimes \mathbb{Z}/4$$

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with $\mathbb{Z}/4$ acting faithfully on $\mathbb{Z}/p$ (thus $p \equiv 1 \mod 4$ in that case). The p-primary cohomology of these groups is given by

$$\begin{aligned} H^*(\Sigma_p;\mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)}[x_{2(p-1)}]/(px_{2(p-1)}),\\ H^*(D_{2p};\mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)}[x_4]/(px_4),\\ H^*(M_{4p};\mathbb{Z}_{(p)}) &= \mathbb{Z}_{(p)}[x_8]/(px_8),\end{aligned}$$

where x_i denotes a generator of dimension i. We can now state our results. By abuse of notation, we shall write nA for the n-fold direct sum of an Abelian group A.

Theorem 1. *Let $p > 2$ be a prime. Then*

$$H^*(Out(F_{p-1});\mathbb{Z}_{(p)}) = \begin{cases} H^*(\Sigma_p;\mathbb{Z}_{(p)}), & \text{if } * > 2p-5\\ H^*(X_{p-1}/Out(F_{p-1});\mathbb{Z}_{(p)}), & \text{if } * \leq 2p-5.\end{cases}$$

Theorem 2. *Let $p > 2$ be a prime. Then*

$$H^*(Out(F_p);\mathbb{Z}_{(p)}) = \begin{cases} 2H^*(\Sigma_p;\mathbb{Z}_{(p)}), & \text{if } * > 2p-3\\ H^*(X_p/Out(F_p);\mathbb{Z}_{(p)}), & \text{if } * \leq 2p-3.\end{cases}$$

Theorem 3. *Let $p > 3$ be a prime. Then*

(1) *if $p \equiv 3 \mod 4$ and $* > 2p-1$,*

$$H^*(Out(F_{p+1});\mathbb{Z}_{(p)}) = 3H^*(\Sigma_p;\mathbb{Z}_{(p)}) \oplus H^*(D_{2p};\mathbb{Z}_{(p)}) \oplus (1-\chi(\mathcal{D}))H^{*-1}(D_{2p};\mathbb{Z}_{(p)})$$

(2) *if $p \equiv 1 \mod 4$ and $* > 2p-1$,*

$$H^*(Out(F_{p+1});\mathbb{Z}_{(p)}) = 3H^*(\Sigma_p;\mathbb{Z}_{(p)}) \oplus H^*(M_{4p};\mathbb{Z}_{(p)}) \oplus (1-\chi(\mathcal{D}))H^{*-1}(D_{2p};\mathbb{Z}_{(p)}).$$

The function $\chi(\mathcal{D})$ is the Euler characteristic of some connected graph defined below, $1-\chi(\mathcal{D})$ its rank and these integers depend only on the prime p (see Corollary 4.7).

Remark. The cohomology classes in degrees $4l-1$ in $H^*(Out(F_{p+1});\mathbb{Z}_{(p)})$ which according to Theorem 3 come up as soon as $p \geq 17$ (see Corollary 4.7) cannot be detected by restricting to a finite subgroup of $Out(F_{p+1})$. Indeed, if $G \subset Out(F_{p+1})$ is any finite subgroup, its p-Sylow subgroup is isomorphic to $\mathbb{Z}/p$ or trivial, thus $H^*(G;\mathbb{Z}_{(p)})$ is 0 in odd dimensions.

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Section 1: The Brown spectral sequence

For the convenience of the reader we recall here a few basic facts about the simplicial spine X_n of outer space; the basic references for this section are [CV] and [SV]. The contractible simplicial complex X_n is the geometric realization of the partially ordered set of marked admissible graphs of rank n. An admissible graph of rank n is a finite connected one-dimensional CW-complex with no vertex of valence less than 3, no separating edge, and fundamental group free of rank n. A marked admissible graph is an equivalence class $[\phi : R_n \to G]$ where R_n is a wedge of n circles, G an admissible graph of rank n and ϕ a homotopy equivalence; the markings $\phi_i : R_n \to G_i$ are considered to be equivalent $(i = 1, 2)$ if there exists a homeomorphism $h : G_1 \to G_2$ such that $h \circ \phi_1$ is homotopic to ϕ_2. Note that we can identify $Out(F_n)$ with $E(R_n)$, the group of free homotopy classes of homotopy equivalences $R_n \to R_n$. One therefore has a natural action of $Out(F_n)$ on marked admissible graph by mapping $[\phi : R_n \to G]$ to $[\phi \circ \lambda : R_n \to G]$ for $\lambda \in Out(F_n)$. The orbits of this action correspond bijectively to homeomorphism types of admissible graphs of rank n. One defines a partial order on the set of marked admissible graphs as follows:

$$[R_n \xrightarrow{\phi_2} G_2] > [R_n \xrightarrow{\phi_1} G_1]$$

if G_2 contains a forest (a disjoint union of trees) such that G_1 is obtained from G_2 by collapsing each tree of the forest to a point, with the collapsing map $G_2 \to G_1$ being compatible with the markings. For instance, for $n = 2$ there are only 2 types of admissible rank 2 graphs, $R_2 = S^1 \vee S^1$ and Θ_2, the graph with 2 vertices and 3 edges connecting them. It follows that X_2 contains no 2-simplices and, as X_n is a contractible space for every n, one infers that X_2 is a tree. Furthermore, $X_2/Out(F_2)$ is homeomorphic to a 1-simplex and the action of $Out(F_2) \cong Gl_2(\mathbb{Z})$ on X_2 yields the familiar decomposition of $Gl_2(\mathbb{Z})$ as an amalgam of finite groups.

Returning to the general case, the action of $Out(F_n)$ on X_n has finite cell stabilizers and yields a spectral sequence (cf. Brown's book [Bro]) of the form

$$(\mathcal{S}) \qquad \prod_{\bar\sigma \in \mathcal{O}(X_n)} H^*(stab(\sigma); \mathbb{Z}_{(p)}) \Rightarrow H^*(Out(F_n); \mathbb{Z}_{(p)}),$$

where $\bar\sigma$ runs over the set of orbits $\mathcal{O}(X_n)$ of simplices $\sigma \subset X_n$. It is known that $\dim(X_n) = 2n - 3$ for $n > 1$ so that the p-primary part of the cohomology of $Out(F_n)$ in dimension $> 2n - 3$ depends only on the stabilizers $stab(\sigma)$ whose order is divisible by p. If the k-simplex σ is determined by vertices

$$[R_n \to G_0] < [R_n \to G_1] < ... < [R_n \to G_k]$$

then $stab(\sigma)$ is naturally isomorphic to the subgroup of $Aut(G_k)$, the group of graph automorphisms of G_k, which is compatible with the chain $F_1 \subset F_2 \subset ... \subset F_k$ of subforests of G_k whose successive collapse yields $G_k \to G_{k-1} \to ... \to G_0$. If $\mathbb{Z}/p \subset stab(\sigma)$, the graphs G_i, $0 \le i \le k$, all admit a $\mathbb{Z}/p$-symmetry compatible with the collapsing maps. In particular

$$stab([R_n \to G]) \cong Aut(G)\,,$$

the group of graph automorphisms of G (see [CV]). To be able to compute cohomology from the spectral sequence above, we will have to study admissible graphs with $\mathbb{Z}/p$-symmetry, and chains of such graphs connected by collapses of $\mathbb{Z}/p$-invariant subforests. The part of X_n relevant for our computations is the p-singular subcomplex X_n^{sing}, consisting of all simplices with stabilizers of order divisible by p. We will repeatedly use the fact that $\mathbb{Z}/p \times \mathbb{Z}/p \not\subset stab(\sigma)$ for the cases we consider (p-rank one case) and that for $n = p-1$, p, $p+1$ and $p > 2$ the group $Out(F_n)$ does not contain an element of order p^2; both these facts follow from well-known facts on torsion elements in $Gl_n(\mathbb{Z})$ (see also [GMV]), using the natural map from $Out(F_n)$ to $Gl_n(\mathbb{Z})$, which is injective on torsion subgroups.

Section 2: The case $n = p - 1$

Let $p > 2$ be a prime. It is easy to check that the only admissible graph type of rank $p-1$ with $\mathbb{Z}/p$-symmetry is the graph Θ_{p-1}, with 2 vertices and p edges connecting them, which are permuted by $\mathbb{Z}/p$. It follows that X_{p-1}^{sing} does not contain any 1-simplices, and therefore $X_{p-1}^{sing} \subset X_{p-1}$ is a discrete subspace, consisting of one orbit $Out(F_{p-1})v$, with $v = [R_{p-1} \to \Theta_{p-1}]$ a particular vertex of X_{p-1}^{sing}. It follows that in the Brown spectral sequence one has

$$E_1^{s,t} = \prod_{\bar{\sigma} \in \mathcal{O}^s(X_{p-1})} H^t(stab(\sigma); \mathbb{Z}_{(p)}) = 0, \qquad \text{for } s \cdot t \neq 0,$$

where $\mathcal{O}^s(X_{p-1})$ denotes the set of orbits of s-simplices of X_{p-1}. Note that the cohomology of the chain complex $(E_1^{*,0}, d_1)$ is $H^*(X_{p-1}/Out(F_{p-1}); \mathbb{Z}_{(p)})$. As observed above, if p divides $|stab(\sigma)|$ then σ is a 0-simplex of type Θ_{p-1} and therefore the whole symmetric group $\Sigma_p \subset Aut(\Theta_{p-1})$ is contained in $stab(\sigma)$; we used here the fact we already mentioned that, for any vertex $v = [R_{p-1} \to G]$, $stab(v)$ is isomorphic to $Aut(G)$. The p-Sylow subgroup of $stab(\sigma)$ is $\mathbb{Z}/p$ (because $Out(F_{p-1})$ has p-rank one, and $\mathbb{Z}/p^2 \not\subset Out(F_{p-1})$), and thus

$$H^*(stab(\sigma); \mathbb{Z}_{(p)}) = H^*(\mathbb{Z}/p; \mathbb{Z}_{(p)})^W$$

where $W \subset Aut(\mathbb{Z}/p)$ is the image of the normalizer $N(\mathbb{Z}/p) \subset stab(\sigma)$ in $Aut(\mathbb{Z}/p)$ under the natural action of $N(\mathbb{Z}/p)$ on $\mathbb{Z}/p$. As $\Sigma_p \subset stab(\sigma)$ and Σ_p contains the holomorph H of a Sylow subgroup $\mathbb{Z}/p \subset \Sigma_p$ (i.e., $H = \mathbb{Z}/p \rtimes Aut(\mathbb{Z}/p)$) we infer that the restriction map

$$H^*(stab(\sigma); \mathbb{Z}_{(p)}) \to H^*(\Sigma_p; \mathbb{Z}_{(p)})$$

is an isomorphism. It is convenient to express this fact by saying that Σ_p *controls the p-primary cohomology of* $stab(\sigma)$. Therefore the Brown spectral sequence takes on the E_2 level the form

$$E_2^{s,t} = \begin{cases} H^t(\Sigma_p; \mathbb{Z}_{(p)}), & \text{if } s = 0, \\ H^s(X_{p-1}/Out(F_{p-1}); \mathbb{Z}_{(p)}), & \text{if } t = 0, \\ 0, & \text{if } s \cdot t \neq 0. \end{cases}$$

Because $H^t(\Sigma_p; \mathbb{Z}_{(p)}) = 0$ for $0 < t < 2(p-1)$ and $H^s(X_{p-1}/Out(F_{p-1}); \mathbb{Z}_{(p)}) = 0$ for $s > 2p-5 = \dim X_{p-1}$, one has $E_2^{s,t} = E_\infty^{s,t}$ in the spectral sequence (recall that the differential d_r has bidegree $(r, 1-r)$). As a result, we obtain Theorem 1.

Section 3: The case $n = p$

Let $p > 2$ be a prime. The reader can easily verify that there are precisely 4 vertex types of rank p with $\mathbb{Z}/p$-symmetry:

- R_p (one vertex, p edges)
- Θ_p (two vertices, $p+1$ edges connecting them)
- W_p (a p-gon plus a vertex connected to the p vertices of the p-gon; W_p has $p+1$ vertices and $2p$ edges)
- $\Theta_{p-1} \vee R_1$ (the one-point union of a graph Θ_{p-1} of rank $p-1$ with a graph R_1, with one vertex and one edge; recall that Θ_{p-1} has 2 vertices and p edges).

Note that the $\mathbb{Z}/p$-action on Θ_p leaves one edge fixed, and that edge defines a $\mathbb{Z}/p$-invariant forest in Θ_p. Collapsing this forest yields a family of 1-simplices

$$[R_p \to \Theta_p] > [R_p \to R_p] .$$

in X_p^{sing}. Similarly, W_p with the obvious $\mathbb{Z}/p$-action, contains a $\mathbb{Z}/p$-invariant forest consisting of an invariant vertex together with the p edges connected to that vertex, yielding another family of 1-simplices

$$[R_p \to W_p] > [R_p \to R_p]$$

As these are the only 1-simplices in X_p^{sing}, it follows that $X_p^{sing}/Out(F_p)$ is a 1-dimensional simplicial complex with 2 components, of the form

$$\overset{(\Theta_{p-1}\vee R_1)}{\bullet} \qquad \overset{(\Theta_p)}{\bullet} \longrightarrow \overset{(R_p)}{\bullet} \longleftarrow \overset{(W_p)}{\bullet}$$

For the p-primary cohomology of the vertex and edge stabilizers one notes again that the groups in question are finite groups of p-rank 1 with p-Sylow subgroup $\mathbb{Z}/p$ (recall that the group $Out(F_p)$ does not contain $\mathbb{Z}/p^2$). It is then straightforward from the symmetries of the graphs R_p, Θ_p, W_p and $\Theta_{p-1} \vee R_1$ that the p-part of the cohomology of the vertex stabilizers is controlled by the following subgroups:

$$\begin{aligned} \Sigma_p &\subset Aut(R_p) , \\ \Sigma_p &\subset Aut(\Theta_p) , \\ D_{2p} &\subset Aut(W_p) , \\ \Sigma_p &\subset Aut(\Theta_{p-1} \vee R_1) . \end{aligned}$$

The case $p = 3$ is somewhat special because W_3 is isomorphic to the 1-skeleton of a 3-simplex and has thus symmetry group Σ_4. But $\Sigma_3 \subset \Sigma_4$ controls 3-primary cohomology and $\Sigma_3 = D_{2p}$ for $p = 3$, thus $D_6 \subset Aut(W_3)$ controls 3-primary cohomology, as claimed. Note that $\Sigma_p \subset Aut(\Theta_p)$ leaves one of the $p+1$ edges of Θ_p fixed so that the collapsing map $\Theta_p \to R_p$ is Σ_p equivariant. Thus Σ_p is contained in the stabilizer of the corresponding 1-simplex and controls the p-primary cohomology. In a similar way does D_{2p} control the p-primary cohomology of the stabilizer of the 1-simplex coming from $W_p \to R_p$. The orbits of simplices under the $Out(F_p)$ action in the singular part X_p^{sing} correspond thus to two components

$$\underset{\Sigma_p}{\bullet} \qquad\qquad \underset{\Sigma_p}{\bullet} \xrightarrow{\Sigma_p} \underset{\Sigma_p}{\bullet} \xleftarrow{D_{2p}} \underset{D_{2p}}{\bullet}$$

where the labels of the edges and vertices indicate the subgroups of the corresponding stabilizers which control p-primary cohomology. Using these pictures, and because the differentials

$$d_1 : E_1^{s,t} \to E_1^{s+1,t}$$

in the Brown spectral sequence correspond to a sum of cohomological restriction maps, we see that

$$E_2^{s,t} = \begin{cases} 2H^t(\Sigma_p; \mathbb{Z}_{(p)}), & \text{for } s = 0,\ t > 0, \\ 0, & \text{for } s > 0,\ t > 0, \\ H^s(X_p/Out(F_p); \mathbb{Z}_{(p)}), & \text{for } t = 0. \end{cases}$$

As in the case $n = p-1$, we note that $\dim X_p = 2p-3 < 2(p-1)$, which implies that all higher differentials vanish for dimension reason, thus $E_2^{s,t} = E_\infty^{s,t}$, and Theorem 2 follows. Brady proved ([Bra]) that $X_3/Out(F_3)$ is contractible. Theorem 2 thus implies that

$$H^*(Out(F_3); \mathbb{Z}_{(3)}) = \begin{cases} \mathbb{Z}_{(3)}, & \text{for } * = 0, \\ 0, & \text{for } * > 0 \text{ and } * \not\equiv 0 \bmod 4, \\ \mathbb{Z}/3 \oplus \mathbb{Z}/3, & \text{for } * > 0 \text{ and } * \equiv 0 \bmod 4. \end{cases}$$

This could also have been deduced from Brady's calculation ([Bra]) of $H^*(Out_+(F_3); \mathbb{Z})$, where $Out_+(F_3) \subset Out(F_3)$ denotes the kernel of the natural composite map

$$Out(F_3) \to Gl_3(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}/2\,.$$

Section 4: The case $n = p + 1$

Recall that the p-rank of $Out(F_n)$ equals 1 precisely if $p-1 \leq n < 2(p-1)$. If G is a connected graph of rank n, the G admits a fixed point free automorphism of order p only if p divides $\chi(G)=n-1$. Thus, the only n in the range $p-1 \leq n < 2(p-1)$ for which there exist (admissible) graphs of rank n with free $\mathbb{Z}/p$-symmetry is $n = p+1$, $p \geq 5$. This existence of admissible graphs with free $\mathbb{Z}/p$-symmetry makes this case $n = p+1$ more complicated, as we will see below.

Let $p \geq 5$ be a prime. We first want to describe the admissible graphs of rank $p+1$ with $\mathbb{Z}/p$-symmetry. We will show that there are precisely

$$12 + (2[\frac{p+3}{4}] + [\frac{p+5}{6}] + 2)$$

such, 12 of which are constructed from graphs we encountered already during the discussion of the rank $p-1$, resp. rank p case, and correspond to graphs for which the $\mathbb{Z}/p$-symmetry has a fixed point, whereas the others belong to new families represented by graphs with fixed point free symmetry. First, we will describe the 12 graphs whose $\mathbb{Z}/p$-symmetry has a fixed point. They are listed with increasing number of vertices. The first three are minimal in the sense that they do not admit any $\mathbb{Z}/p$-equivariant collapsing map to a smaller example. The indices of graphs always refer to their rank.

- R_{p+1} (one vertex, $p+1$ edges — a rose with p+1 leaves)
- $\Theta_{p-1} \vee R_2$ (one-point union of Θ_{p-1} and R_2)
- $R_1 \vee \Theta_{p-1} \vee R_1$ (Θ_{p-1} with each of the two vertices identified with a vertex of a copy of the graph R_1)
- Θ_{p+1} (two vertices, connected by $p+2$ edges)
- $\Theta_p \vee R_1$ (Θ_p one-point union R_1)
- $\Theta_p * R_1$ (Θ_p with one edge subdivided to get a new vertex, which is identified with the vertex of the circle R_1)
- $\Theta_{p-1} \vee \Theta_2$ (one-point union of Θ_{p-1} and Θ_2)
- $\Theta_{p-1} \vee \Theta_1 \vee R_1$ (with the vertex of R_1 being identified with the vertex of valence 2 of $\Theta_{p-1} \vee \Theta_1$)
- $\Theta_{p-1} \diamond Y$ (a graph Θ_{p-1} connected to a third vertex by means of three edges, two connecting one vertex of Θ_{p-1} to the new vertex, and the third edge connecting the other vertex of Θ_{p-1} to the new vertex)
- $\Theta_{p-1} * \Theta_2$ (a graph Θ_{p-1} with one of its vertices connected with the the midpoint of an edge of Θ_2 yielding a graph with four vertices)
- $\Theta_{p-1} ** \Theta_1$ (a graph Θ_{p-1} joined to a graph Θ_1 by one vertex of Θ_{p-1} to one of Θ_1, and the other edge connecting the two remaining vertices)
- $W_p \vee R_1$ (the graph W_p with its central vertex identified with the vertex of a copy of R_1)

Note that that each of these 12 graphs has an obvious $\mathbb{Z}/p$-symmetry and each of these has a unique $\mathbb{Z}/p$-invariant maximal subtree up to a $\mathbb{Z}/p$-equivariant graph automorphism, with the exception of $\Theta_{p-1} * \Theta_2$, which has two such maximal subtrees. They give rise to 12 vertices in $X_{p+1}^{sing}/Out(F_{p+1})$, belonging to 3 different connected components which correspond to the three "minimal graphs" of our list. Our goal is to prove that these three connected components are contractible. They look schematically as follows. The first one, called $\mathcal{A}$, is

$$(\mathcal{A})\qquad \begin{array}{ccccccc} \Theta_{p-1} ** \Theta_1 & \longrightarrow & \Theta_{p-1} \diamond Y & \longrightarrow & \Theta_{p+1} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \Theta_p * R_1 & \longrightarrow & \Theta_p \vee R_1 & \longrightarrow & R_{p+1} & \longleftarrow & W_p \vee R_1 \end{array}$$

As a simplicial complex this component is a one-point union of a 3-dimensional complex $\mathcal{A}(3)$ and an edge, corresponding to $R_{p+1} \leftarrow W_p \vee R_1$. The complex $\mathcal{A}(3)$ is a cone over the pentagon formed by the three adjacent triangles with vertices $(\Theta_{p-1} \diamond Y, \Theta_{p+1}, R_{p+1})$, $(\Theta_{p-1} \diamond Y, \Theta_p \vee R_1, R_{p+1})$, and $(\Theta_p * R_1, \Theta_p \vee R_1, R_{p+1})$. This implies that $\mathcal{A}$ is indeed contractible. The point is that the different edges between corresponding vertices, which arise from the collapsing of two different $\mathbb{Z}/p$-invariant trees, e.g.

$$\Theta_{p-1} ** \Theta_1 \rightrightarrows \Theta_p \vee R_1,$$

become one single edge in $X_{p+1}^{sing}/Out(F_{p+1})$, because there is a $\mathbb{Z}/p$-equivariant symmetry of $\Theta_{p-1} * *\Theta_1$ interchanging the two trees. Similar for the simplices of dimension two and three.

The second component $\mathcal{B}$ is 2-dimensional and has the following shape.

$$(\mathcal{B}) \qquad \begin{array}{ccc} \Theta_{p-1} * \Theta_2 & \longrightarrow & \Theta_{p-1} \vee \Theta_1 \vee R_1 \\ \downarrow & & \downarrow \\ \Theta_{p-1} \vee \Theta_2 & \longrightarrow & \Theta_{p-1} \vee R_2 \end{array}$$

There are two different edges e_1 and e_2 joining $\Theta_{p-1} * \Theta_2$ to the vertex $\Theta_{p-1} \vee R_2$, stemming from collapsing the two different kinds of maximal subtrees in $\Theta_{p-1} * \Theta_2$. Looking carefully at the 2-simplices one sees that $\mathcal{B}$ consists of a cone over the circle $e_1 \cup e_2$, with an additional cone over one of the edges e_i attached to it, which makes $\mathcal{B}$ contractible.

The third component $\mathcal{C}$ is a single point.

$$(\mathcal{C}) \qquad \bullet \quad R_1 \vee \Theta_{p-1} \vee R_1$$

For these three components $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the contribution to the E_2-term of the Brown spectral sequence can easily be computed as follows. In $(\mathcal{A})$, we can replace the part of the diagram

$$\begin{array}{ccccc} \Theta_{p-1} * *\Theta_1 & \longrightarrow & \Theta_{p-1} \diamond Y & \longrightarrow & \Theta_{p+1} \\ \downarrow & & \downarrow & & \downarrow \\ \Theta_p * R_1 & \longrightarrow & \Theta_p \vee R_1 & \longrightarrow & R_{p+1}, \end{array}$$

which corresponds to two 3-simplices with a common vertex, by that vertex at the right hand corner

$$\bullet \quad R_{p+1}\,.$$

This can be seen as follows. First, the automorphism group of each vertex of that complex has p-primary cohomology controlled by Σ_p, same for all higher simplices, as all collapsing maps are Σ_p-equivariant. Second, one uses the following lemma.

Lemma 4.1. *Let ϕ be an automorphism of Σ_p. Then the induced map in p-primary cohomology*

$$\phi^* : H^*(\Sigma_p; \mathbb{Z})_{(p)} \to H^*(\Sigma_p; \mathbb{Z})_{(p)}$$

is the identity map.

Proof. Let α be an element of order p in Σ_p and consider the subgroups G and H of Σ_p generated by α and $\phi(\alpha)$ respectively. Choose $\beta \in \Sigma_p$ such that $\beta H \beta^{-1} = G$, thus $g \mapsto \beta\phi(g)\beta^{-1}$ maps G to G; such a β exists, because G and H are p-Sylow subgroups of Σ_p. Because conjugation by any element in Σ_p induces the identity in cohomology, we may therefore assume that $G = H$. But the restriction map

$$\mathrm{res}(\Sigma_p, G) : H^*(\Sigma_p; \mathbb{Z})_{(p)} \to H^*(G; \mathbb{Z})$$

is injective, and

$$H^*(G;\mathbb{Z})) = \mathbb{Z}[x]/(px)$$

for some $x \in H^2$. It follows that ϕ restricted to G induces the identity in $H^{2(p-1)}(G;\mathbb{Z})$, and the the proof of the Lemma is completed by observing that the restriction map $\operatorname{res}(\Sigma_p, G)$ maps into the subalgebra generated by x^{p-1}.

As we will see, $X_{p+1}^{sing}/Out(F_{p+1})$ has altogether four connected components $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and $\mathcal{D}$, with the last component $\mathcal{D}$ made up from graphs with a fixed point free $\mathbb{Z}/p$ symmetry. Accordingly, the E_1-term of the spectral sequence $(\mathcal{S})$ decomposes in total degree larger than the virtual cohomological dimension of $Out(F_{p+1})$ into four terms

$$(\mathcal{A})^{s,t} \oplus (\mathcal{B})^{s,t} \oplus (\mathcal{C})^{s,t} \oplus (\mathcal{D})^{s,t} = E_1^{s,t}, \quad s+t > 2p-1.$$

Passing to the E_2-term, our discussion above shows that, in the range $s+t > 2p-1$,

$$H(\mathcal{A})^{s,t} \cong H(\mathcal{B})^{s,t} \cong H(\mathcal{C})^{s,t} \cong \begin{cases} 0, & \text{if } t > 0 \\ H^s(\Sigma_p;\mathbb{Z}_{(p)}), & \text{if } s \geq 0,\ t = 0. \end{cases}$$

Therefore, in our Theorem 3 of the introduction, we obtain from the components $(\mathcal{A})$, $(\mathcal{B})$ and $(\mathcal{C})$, the contribution $3H^*(\Sigma_p;\mathbb{Z}_{(p)})$ to $H^*(Out(F_{p+1});\mathbb{Z}_{(p)})$. It remains to compute the contribution of the last component $\mathcal{D}$. The vertices of $\mathcal{D}$ arise from 3 families of graphs $\{P_k\}$, $\{S_k\}$ and $\{T_k\}$, where $0 \leq k \leq p-1$, and $k > 0$ in case of P_k, with p a fixed prime ≥ 5. The number $v(p)$ of isomorphism types of such graphs is a rather complicated function of p because in these families we will have to identify isomorphic graphs. All these graphs are characterized by the fact that they are admissible and admit a $\mathbb{Z}/p$ action with no fixed point. Each of them has therefore a number of vertices and edges which is divisible by p, say np and mp, so that

$$np - mp = -p\,,$$

where $-p$ is the Euler characteristic of a rank $p+1$ graph. Since in an admissible graph all vertices have valency at least 3, one has $3np \leq 2mp$ thus $2np - 2mp = -2p \leq -np$, which implies $n \leq 2$. The minimal possible n is 1 and leads to the family $\{T_k\}$ with T_k defined by its vertices $v_0, \dots, v_{p-1}$ and edges $e_0, f_0, \dots, e_{p-1}, f_{p-1}$, having endpoints

$$\partial e_i = \{v_i, v_{i+1}\}, \quad \partial f_i = \{v_i, v_{i+k}\}\,.$$

These graphs have an obvious $\mathbb{Z}/p$-symmetry, given by

$$v_i \mapsto v_{i+1}, \quad e_i \mapsto e_{i+1}, \quad f_i \mapsto f_{i+1}\,.$$

Note also that the $\mathbb{Z}/p$-action is essentially unique, because the automorphism group of T_k has p-Sylow group isomorphic to $\mathbb{Z}/p$ (p-rank 1 case!), thus all possible $\mathbb{Z}/p$-actions are conjugate; the same applies to the other two families we will define. There is also a natural involution on these T-graphs given by

$$\theta_T(v_i) = v_{-i}, \quad \theta_T(e_i) = e_{-i-1}, \quad \theta_T(f_i) = f_{i-k}\,.$$

This involution, together with the $\mathbb{Z}/p$-symmetry, generates what we will call the *dihedral* subgroup $D_{2p} \subset Aut(T_k)$.

It is easy to see that $T_k \cong T_l$ as graphs if and only if $k = \pm l$ or $kl = \pm 1 \mod p$. The isomorphism types of T's correspond thus to the orbits of a $\mathbb{Z}/2 \times \mathbb{Z}/2$-action on the set $\{0, 1, \dots, p-1\}$, with orbits depending on $p \mod 4$. If $p \equiv 3 \mod 4$ one has orbits $\{0\} \coprod \{1, p-1\} \coprod \{\text{orbits of length } 4\}$, thus $2 + (p-3)/4$ orbits, and in that case $p \equiv 1 \mod 4$ one has orbits $\{0\} \coprod \{1, p-1\} \coprod \{\omega, -\omega\}$ and the rest orbits of length 4, thus $3 + (p-5)/4$. Here $\pm\omega$ are the square roots of $-1 \mod p$. If we write $[z]$ for the integral part of the rational number z then we can express our computation as follows.

Lemma 4.2. *The number $v_T(p)$ of isomorphism types of graphs $\{T_k\}, 0 \leq k < p$, is given by*

$$v_T(p) = [\frac{p+7}{4}].$$

The graphs T_k exhausts all rank $p+1$ graphs with p vertices and $\mathbb{Z}/p$-symmetry. The graphs with $2p$ vertices come in two families, the first one denoted by $\{P_k\}$, $0 < k < p$, which is defined as follows. Each graph P_k has $2p$ vertices $\tilde{v}_0, \tilde{w}_0, \dots, \tilde{v}_{p-1}, \tilde{w}_{p-1}$ and $3p$ edges $\tilde{e}_i, \tilde{f}_i, \tilde{g}_i$ with $0 \leq i < p$, having endpoints

$$\partial \tilde{e}_i = \{\tilde{v}_i, \tilde{v}_{i+1}\}, \quad \partial \tilde{f}_{ki} = \{\tilde{w}_i, \tilde{w}_{i+1}\}, \quad \partial \tilde{g}_i = \{\tilde{v}_{ik}, \tilde{w}_i\}.$$

The indices in the definition above are always understood to be taken modulo p, and k^{-1} stands for the inverse of $k \mod p$. Note that $k \neq 0$ for the family $\{P_k\}$. The graphs P_k have a $\mathbb{Z}/p$-symmetry, given by

$$\tilde{v}_i \mapsto \tilde{v}_{i+1}, \quad \tilde{w}_i \mapsto \tilde{w}_{i+k^{-1}}, \quad \tilde{e}_i \mapsto \tilde{e}_{i+1}, \quad \tilde{f}_i \mapsto \tilde{f}_{i+1}, \quad \tilde{g}_i \mapsto \tilde{g}_{i+k^{-1}}.$$

Again, there is an involution of these P-graphs given by

$$\theta_P(\tilde{v}_i) = \tilde{v}_{-i},\ \theta_P(\tilde{w}_i) = \tilde{w}_{-i},\ \theta_P(\tilde{e}_i) = \tilde{e}_{-i-1},\ \theta_P(\tilde{f}_i) = \tilde{f}_{-i-k},\ \theta_P(\tilde{g}_i) = \tilde{g}_{-i}.$$

This involution, together with the $\mathbb{Z}/p$-symmetry, generates the *dihedral* subgroup $D_{2p} \subset Aut(P_k)$. We can count the isomorphism types as before, using that $P_k \cong P_l$ if and only if $k = \pm l$ or $k = \pm l^{-1}$ (indices mod p). We thus have the following.

Lemma 4.3. *The number $v_P(p)$ of isomorphism types of graphs $\{P_k\}$, $0 < k < p$, is given by*

$$v_P(p) = [\frac{p+3}{4}].$$

The third family $\{S_k\}$, $0 \leq k < p$, is defined as follows. The graph S_k has $2p$ vertices $a_0, b_0, \dots, a_{p-1}, b_{p-1}$ and $3p$ edges x_i, y_i, z_i with $0 \leq i < p$, having endpoints

$$\partial x_i = \{a_i, b_i\}, \quad \partial y_i = \{b_i, a_{i+1}\}, \quad \partial z_i = \{a_i, b_{i+k}\}.$$

These graphs have an obvious $\mathbb{Z}/p$-symmetry, given by

$$a_i \mapsto a_{i+1}, \quad b_i \mapsto b_{i+1}, \quad x_i \mapsto x_{i+1}, \quad y_i \mapsto y_{i+1}, \quad z_i \mapsto z_{i+1},$$

and a symmetry θ_S of order 2, which we call again the dihedral symmetry,

$$\theta_S(a_i) = b_{-i}, \quad \theta_S(b_{-i}) = a_i \,.$$

These two symmetries generate a subgroup $D_{2p} \subset Aut(S_k)$, which we call the *dihedral* subgroup. The group D_{2p} acts transitively on the vertices of S_k. Next, we have to see how the graphs split into isomorphism classes. If we have an isomorphism $S_k \to S_l$, we can assume that a_0 is mapped to a_0 by composing with a D_{2p}-symmetry if necessary. One finds then easily the following basic isomorphism

$$\alpha : S_k \mapsto S_{-k-1}, \quad (a_i \mapsto a_{-i}\,, b_i \mapsto b_{-i-1})\,,$$

and, in case $k \neq 0$

$$\beta : S_k \mapsto S_{k^{-1}}, \quad (a_i \mapsto a_{-ik^{-1}}, b_i \mapsto b_{-ik^{-1}})\,.$$

It is not hard to see that (up to the $\mathbb{Z}/p$ symmetries) these maps α and β generate all the isomorphisms between these graphs. Thus, to divide them into isomorphism classes, we have to partition the indexing set $\{0, 1, \ldots p-1\}$ into orbits with respect to the induced action (which we also denote by α and β !) on this indexing set: $\alpha(k) = -k-1$, $\beta(k) = k^{-1}$ if $k \neq 0$ and, for convenience, $\beta(0) = 0$. For instance $\{0, p-1\}$ is an orbit, and on the remaining part $\{1, 2, 3, \ldots, p-2\}$ the two involutions α and β generate a group isomorphic to Σ_3 (because for $k \neq 0, -1$ one has $\alpha\beta\alpha(S_k)=\beta\alpha\beta(S_k)$), and the orbits therefore have length $1, 2, 3$ or 6. A simple calculation shows that the orbit decomposition is

$$\{0,1,2,\ldots,p-1\} = \{0,p-1\} \coprod \{1, \frac{p-1}{2}, p-2\} \coprod \{\lambda, \mu\} \coprod \{\text{orbits of leghth } 6\}\,,$$

where λ and μ are the two solutions of

$$k^2 + k + 1 = 0 \mod p, \quad p > 3\,.$$

This equation has solutions mod p if and only if $\sqrt{-3} \in \mathbb{F}_p$. Using quadratic reciprocity it is easy to see that for $p > 3$ this is equivalent to $p \equiv 1 \mod 3$. We thus have the following.

Lemma 4.4. *If $v_S(p)$ denotes the number of isomorphism types of graphs $\{S_k\}$, where $0 \leq k < p$, then*

$$v_S(p) = [\frac{p+11}{6}] = \begin{cases} 3 + (p-7)/6, & \text{if } p \equiv 1 \mod 3 \\ 2 + (p-5)/6, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Corollary 4.5. *The number of vertices $v(p) \in X^{sing}_{p+1}/Out(F_{p+1})$ corresponding to graphs with a fixed point free $\mathbb{Z}/p$-action is*

$$v(p) = v_P(p) + v_S(p) + v_T(p) = [\frac{p+3}{4}] + [\frac{p+11}{6}] + [\frac{p+7}{4}] = 2[\frac{p+3}{4}] + [\frac{p+5}{6}] + 2.$$

There are $\mathbb{Z}/p$-equivariant collapsing maps

$$P_k \to T_k, \quad 0 < k \leq p-1,$$

by collapsing the unique $\mathbb{Z}/p$-invariant forest of P_k. Similarly, there are collapsing maps

$$S_k \to T_k, \quad 0 \leq k \leq p-1, \quad (a_i \mapsto v_i, b_i \mapsto v_i, \quad 0 \leq i \leq p-1)$$

and

$$S_{k-1} \to T_k, \quad 0 \leq k \leq p-1, \quad (a_i \mapsto v_i, b_{i-1} \mapsto v_i, \quad 0 \leq i \leq p-1)$$

and

$$S_{(k-1)^{-1}} \to T_k, \quad k \neq 1 \text{ and } \ 0 \leq k < p, \quad (a_i \mapsto v_{i(k-1)}, \quad b_{i+(k-1)^{-1}} \mapsto v_{i(k-1)}).$$

In particular, the only collapsing maps ending at T_0 are

$$S_0 \to T_0, \quad S_{-1} \to T_0,$$

giving rise to one edge in $X_{p+1}^{sing}/Out(F_{p+1})$

$$\{S_0, S_{-1}\} \text{——} \{T_0\}.$$

Similarly, if $k = \pm 1$, the collapsing maps ending at $T_{\pm 1}$ are

$$S_1 \to T_1, \quad S_0 \to T_1, \quad S_{-1} \to T_{-1}, \quad S_{-2} \to T_{-1}, \quad S_{(-2)^{-1}} \to T_{-1}$$

giving rise to 2 edges

$$\{S_0, S_{-1}\} \text{——} \{T_1, T_{-1}\} \text{——} \{S_1, S_{\frac{p-1}{2}}, S_{p-2}\}.$$

In general, if $k \neq 0, \pm 1$, the 12 S-candidates which could collapse to a member of a T-orbit, $\{T_k, T_{-k}, T_{k^{-1}}, T_{-k^{-1}}\}$, are given by

$$S_k, S_{k-1}, S_{(k-1)^{-1}}; \quad S_{-k}, S_{-k-1}, S_{(-k-1)^{-1}};$$

$$S_{k^{-1}}, S_{k^{-1}-1}, S_{(k^{-1}-1)^{-1}}; \quad S_{-k^{-1}}, S_{-k^{-1}-1}, S_{(-k^{-1}-1)^{-1}}.$$

They fall into (at most) two S-orbits:

$$\{S_k, S_{-k-1}, S_{(-k-1)^{-1}}, S_{(k+1)^{-1}-1}, S_{-1-k^{-1}}, S_{k^{-1}}\},$$

$$\{S_{-k}, S_{k-1}, S_{(k-1)^{-1}}, S_{(-k+1)^{-1}-1}, S_{-1+k^{-1}}, S_{-k^{-1}}\}.$$

It follows that $X_{p+1}^{sing}/Out(F_{p+1})$ has a 1-dimensional connected component, which we call $\mathcal{D}$, with $v(p)$ vertices. We call $\mathcal{D}$ the *free* component, because its vertices correspond to graphs with free $\mathbb{Z}/p$-action. Since every vertex of $\mathcal{D}$ is connected by an edge to a vertex of type T, one can compute the number $e(p)$ of edges of $\mathcal{D}$ by determining the valencies of the T-vertices. For instance, $\overline{T_0} \in \mathcal{D}$ has valency 1; it is connected to $\overline{S_0} = \overline{S_{-1}}$. For $k \neq 0$ the vertex $\overline{T_k} \in \mathcal{D}$ is connected to $\overline{S_k}, \overline{S_{k-1}} (= \overline{S_{-k^{-1}}} = \overline{S_{-k}})$ and $\overline{P_k}$, so its valence $\nu(k)$ is either 2 or 3 depending on whether $\overline{S_k} = \overline{S_{k-1}}$ or not. But $\overline{S_k} = \overline{S_{k-1}}$ if and only if k is either a solution of

$$(*) \qquad k^2 \pm k = 1 \mod p,$$

or of

$$(**) \qquad k^2 = -1 \mod p.$$

For $p > 5$ the equation $(*)$ has a solution if and only if $p \equiv 1,4 \mod 5$, as one easily checks (using quadratic reciprocity). If it has a solution k_1, the other solution k_2 satisfies $k_1 \cdot k_2 = 1 \mod p$ so that T_{k_1} and T_{k_2} lie in the same T-orbit. The equation $(**)$ has a solution if and only if $p \equiv 1 \mod 4$, and if k_3 is a solution, then the other solution k_4 is $-k_3$ so that again T_{k_3} and T_{k_4} lie in the same T-orbit. Note also that for $p > 5$, a solution of $(**)$ can never be a solution of $(*)$. This analysis shows the following.

Corollary 4.6. *The valency $\nu(k)$ of the vertex $\overline{T_k}$ of $\mathcal{D}$ is, for $p > 5$ given by*

(1) $\nu(0) = 1$,
(2) $\nu(k) = 2$ *if* $k^2 \pm k = 1 \mod p$ *or* $k^2 = -1 \mod p$,
(3) $\nu(k) = 3$ *if* $k^2 \pm k \neq 1 \mod p$ *and* $k^2 \neq -1 \mod p$.

Since every 1-simplex of $\mathcal{D}$ has a $\overline{T_k}$ as one of its endpoints, the number $e(p)$ of edges of $\mathcal{D}$ is now easily computed:

$$e(p) = \begin{cases} 3v_T(p) - 2, & \text{if } p \equiv 3,7 \mod 20, \\ 3v_T(p) - 3, & \text{if } p \equiv 11,13,17,19 \mod 20, \\ 3v_T(p) - 4, & \text{if } p \equiv 1,9 \mod 20. \end{cases}$$

Thus, the Euler characteristic of $\mathcal{D}$ is for $p \equiv 3,7 \mod 20$ given by

$$\chi(\mathcal{D}) = v(p) - e(p) = [\frac{p+5}{6}] - [\frac{p+3}{4}] + 1,$$

and similarly for the other cases.

Corollary 4.7. *The component $\mathcal{D}$ has the homotopy type of a wedge of circles. It is contractible if $p < 17$. For $p \geq 5$ its Euler characteristic is given by*

$$\chi(\mathcal{D}) = [\frac{p+5}{6}] - [\frac{p+3}{4}] + \epsilon(p),$$

where

$$\epsilon(p) = \begin{cases} 1, & \text{if } p \equiv 3,7 \mod 20, \\ 2, & \text{if } p \equiv 5,11,13,17,19 \mod 20, \\ 3, & \text{if } p \equiv 1,9 \mod 20. \end{cases}$$

Proof. The formula for $\chi(\mathcal{D})$ follows from our computation of $v(p)$ and $e(p)$, $p > 5$. The case $p = 5$ has to be treated separately. One finds 3 T-orbits

$$\{T_0\} \coprod \{T_1, T_4\} \coprod \{T_2, T_3\}$$

with $\overline{T_0}$ of valency 1, $\overline{T_1} = \overline{T_4}$ of valency 3 and $\overline{T_2} = \overline{T_3}$ of valency 2 (because $2^2 \equiv -1 \mod 4$). The S-orbits are

$$\{S_0, S_4\} \coprod \{S_1, S_2, S_3\}$$

and the P-orbits

$$\{P_1, P_4\} \coprod \{P_2, P_3\}.$$

Thus $\mathcal{D}$ has 7 vertices and $1 + 2 + 3 = 6$ edges, so that $\chi(\mathcal{D}) = 1$ in that case, and therefore $\mathcal{D}$ is contractible.

We need now to compute the p-primary cohomology of the automorphism groups of the graphs of type P, S and T. Because we are in the p-periodic case, the normalizer $N(\mathbb{Z}/p)$ controls p-primary cohomology. Recall that

$$D_{2p} \subset N(\mathbb{Z}/p)$$

and, as we will see, in most cases this subgroup controls p-primary cohomology. First, we will consider the graphs of type P.

Lemma 4.8. *Let $0 < k < p$, and p a prime ≥ 5. Then*

$$H^*(Aut(P_k); \mathbb{Z}_{(p)}) \cong H^*(D_{2p}; \mathbb{Z}_{(p)}).$$

Proof. In our notation, the standard generator ϕ of order p in $Aut(P_k)$ acts by

$$\phi(\tilde{v}_i) = \tilde{v}_{i+1}, \quad \phi(\tilde{w}_i) = \tilde{w}_{i+1}.$$

It follows that every $x \in N(<\phi>)$, the normalizer of the subgroup $<\phi>$ in $Aut(P_k)$, acts on P_k by preserving the $\tilde{v}$-vertices and the $\tilde{w}$-vertices. Note that

$$|N(<\phi>)| = p \cdot |stab_N(\tilde{v}_0)|,$$

where $stab_N(\tilde{v}_0)$ denotes the stabilizer of $\tilde{v}_0 \in P_k$ in $N(<\phi>)$. We want to show that $stab_N(\tilde{v}_0) =<\theta_P>$, θ_P the dihedral reflection (defined earlier). Indeed, if $x \in stab_N(\tilde{v}_0)$ is an element of prime order $q > 2$, it has to act trivially on the 3 edges originating at $\tilde{v}_0$, because two end up at a $\tilde{v}$-vertex and one at a $\tilde{w}$-vertex. But then, as an x normalizes $<\phi>$, it will fix all $\tilde{v}$-vertices: it fixes the neighbors $\tilde{v}_{-1}$ and $\tilde{v}_1$ of $\tilde{v}_0$, and by induction it will then fix all $\tilde{v}$-vertices. Because the edge $\{\tilde{v}_0, \tilde{w}_0\}$ is fixed too, $\tilde{w}_0$ is fixed and it follows by the same reasoning that $x(\tilde{w}_i) = \tilde{w}_i$, for all i, i.e., $x = Id$. On the other hand, if $y \in stab_N(\tilde{v}_0)$ has order 2, one must have $y(\tilde{v}_1) = \tilde{v}_{-1}$, otherwise y would act trivially on the edges originating at $\tilde{v}_0$ and one had, as before, $y = Id$. Thus, if θ_P denotes the dihedral reflection, $\theta_P \cdot y$ acts trivially on the edges originating at $\tilde{v}_0$, and we conclude $\theta_P \cdot y = Id$, which implies $\theta_P = y$, and the proof of the Lemma is completed.

The case of the S-graphs follows.

Lemma 4.9. *Let $0 \leq k < p$ with p a prime ≥ 5. Then*

$$H^*(Aut(S_k);\mathbb{Z}_{(p)}) = H^*(D_{2p};\mathbb{Z}_{(p)}).$$

Proof. Because the dihedral subgroup $D_{2p} \subset N(\mathbb{Z}/p)$ acts transitively on the vertices of S_k, we have

$$|N(\mathbb{Z}/p)| = 2p \cdot |stab_N(a_0)|$$

and it suffices to check that every element of $stab_N(a_0)$ commutes with the $\mathbb{Z}/p$-action, generated by

$$\phi(a_i) = a_{i+1}, \quad \phi(b_i) = b_{i+1}.$$

It then follows that

$$H^*(N(\mathbb{Z}/p);\mathbb{Z}_{(p)}) = H^*(D_{2p};\mathbb{Z}_{(p)}).$$

First, we check that $stab_N(a_0)$ injects into Σ_3, the group of permutations of the 3 edges originating at a_0. Indeed, if $x \in stab_N(a_0)$ fixes all these 3 edges, it fixes the neighbors b_{-1} and b_1. In case x has prime order ≥ 3, we can conclude that x fixes all edges originating at b_{-1}, resp. b_1, and, continuing this way, one sees that $x = Id$, a contradiction. Thus x has order ≤ 2. If it does not commute with ϕ, we must have $x\phi x^{-1} = \phi^{-1}$, thus $xb_i = x\phi^i b_0 = \phi^{-i}b_0 = b_{-i}$, which contradicts the assumption that x fixes b_{-1}, which is a neighbor of a_0. It follows that $stab_N(a_0)$ is isomorphic to a subgroup of Σ_3. If $stab_N(a_0) = \Sigma_3$ then the elements of order 3 in $stab_N(a_0)$ must centralize $< \phi >$, because there is no surjection $\Sigma_3 \to \mathbb{Z}/3 \subset Aut(\mathbb{Z}/p)$. It follows then that $D_{2p} \subset N(\mathbb{Z}/p)$ controls p-primary cohomology. The only case left to consider is the case $stab_N(a_0) = \mathbb{Z}/3$ so that $|N(\mathbb{Z}/p)| = 6p$. If the elements of order 3 in $stab_N(a_0)$ commute with ϕ, it follows again that $D_{2p} \subset N(\mathbb{Z}/p)$ controls the p-primary cohomology. In the other case, the natural map $N(\mathbb{Z}/p) \to Hol(\mathbb{Z}/p) = \mathbb{Z}/p \rtimes \mathbb{Z}/(p-1)$ contains an element w of order 6 whose cube $w^3 = \theta_S$, the dihedral reflection, so that $w^2 = z$ is of order 3 and commutes with θ_S. But θ_S leaves only two edges invariant, namely $\{a_0, b_0\}$ and $\{a_{(p-1)/2}, b_{(p+1)/2}\}$. Thus z fixes a_0 and b_0 therefore all edges originating at a_0. As seen earlier, this forces $z = Id$, a contradiction. It follows that

$$im(N(\mathbb{Z}/p) \to Hol(\mathbb{Z}/p)) = D_{2p}$$

in all cases, which completes the proof of the Lemma.

It remains to consider the case of T graphs. The symmetry group $Aut(T_k)$ contains a copy of the dihedral group D_{2p} of order $2p$, coming from the usual action of D_{2p} on the p-gon formed by the vertices of T_k. This subgroup controls again the p-primary cohomology of $Aut(T_k)$ in most of the cases. But, as we will see, in case k is a primitive fourth root of 1 modulo p, $Aut(T_k)$ contains a symmetry of order 4, which needs to be taken into account. We write as in the introduction M_{4p} for the metacyclic group of order $4p$, with normal subgroup of index 4, on which the factor group $\mathbb{Z}/4$ acts faithfully via conjugation.

Proposition 4.10. *Let $p \geq 5$ be a prime and $0 \leq k < p$.*

(1) *If* $p \equiv 1 \mod 4$,

$$H^*(Aut(T_k);\mathbb{Z}_{(p)}) = \begin{cases} H^*(D_{2p};\mathbb{Z}_{(p)}), & \text{if } k^2 \not\equiv -1 \mod p \\ H^*(M_{4p};\mathbb{Z}_{(p)}), & \text{if } k^2 \equiv -1 \mod p \end{cases}$$

(2) *If* $p \equiv 3 \mod 4$,

$$H^*(Aut(T_k);\mathbb{Z}_{(p)}) = H^*(D_{2p};\mathbb{Z}_{(p)}), \quad \text{for all } k.$$

Proof. Lets first assume that $k \neq 0$. The normalizer $N(\mathbb{Z}/p)$ of $\mathbb{Z}/p$ in $Aut(T_{p,k})$ controls p-primary cohomology. All that really matters is the action of $N(\mathbb{Z}/p)$ on $\mathbb{Z}/p$; we shall write $C \subset Aut(\mathbb{Z}/p)$ for the image of $N(\mathbb{Z}/p)$ with respect to this action. Note that C is a cyclic group of order prime to p. It follows that

$$H^*(Aut(T_k);\mathbb{Z}_{(p)}) = H^*(\mathbb{Z}/p;\mathbb{Z}_{(p)})^C .$$

We will show that there are only two possibilities for C, namely $\mathbb{Z}/2$ resp. $\mathbb{Z}/4$. Accordingly, $Aut(T_k)$ has periodic p-primary cohomology of period 4 resp. 8. We fix a subgroup $\mathbb{Z}/p \subset Aut(T_k)$ by putting $\mathbb{Z}/p =< \phi >$ where ϕ is defined by

$$\phi(v_i) = v_{i+1}, \quad \phi(e_i) = e_{i+1}, \quad \phi(f_i) = f_{i+1}.$$

Because $< \phi >$ and thus $N(< \phi >)$ acts transitively on the vertices of T_k, one has

$$|N(< \phi >)| = p \cdot |stab_N(v_0)|$$

where $stab_N(v_0) \subset N(< \phi >)$ denotes the stabilizer. Note also that, moreover, the inclusion of $stab_N(v_o)$ in $N(< \phi >)$ induces an isomorphism

$$stab_N(v_0) \cong N(< \phi >)/ < \phi > .$$

We claim that $stab_N(v_0)$ admits an embedding into Σ_4, the group of permutations of the 4 edges $\{e_0, e_{p-1}, f_0, f_{p-k}\}$ which have one of their endpoints at v_0. For this, we need to check that if $\psi \in stab_N(v_0)$ fixes these 4 edges, then $\psi = \mathrm{Id}$. Indeed, it is immediate that if ψ fixes all edges originating at v_0, it also fixes v_1 (and v_{p-1}, v_k, v_{p-k}). But then

$$v_1 = \psi(v_1) = \psi \circ \phi(v_0) = \psi \circ \phi \circ \psi^{-1}(v_0) = \phi^l(v_0) = v_l,$$

where $\psi \circ \phi \circ \psi^{-1} = \phi^l$ for some l, because ψ normalizes $< \phi >$. We conclude that $l = 1$, thus $\psi \circ \phi \circ \psi^{-1} = \phi$, which implies $\psi(v_j) = \psi \circ \phi^j(v_0) = \phi^j \circ \psi(v_0) = v_j$, and therefore ψ fixes all vertices. Similarly, one checks then that ψ also fixes all edges, thus $\psi = \mathrm{Id}$. Now, the action $N(\mathbb{Z}/p) \to Aut(\mathbb{Z}/p)$ of $N(< \phi >)$ on $< \phi >$ factors through $N(\mathbb{Z}/p)/\mathbb{Z}/p$, which is isomorphic to $stab_N(v_0)$, and the image $C \subset Aut(\mathbb{Z}/p)$, which is a cyclic group, is therefore a cyclic sub-quotient of Σ_4; the only possibilities for C are thus $\{1\}$, $\mathbb{Z}/2$, $\mathbb{Z}/3$ or $\mathbb{Z}/4$. To rule out the trivial group, we observe that C contains an involution, namely the image of the dihedral involution θ_T. Thus $C = \mathbb{Z}/2$ or $\mathbb{Z}/4$. Next, we show that if C contains an element of order 4, then necessarily $k^2 \equiv -1 \mod p$, and in particular $p \equiv 1 \mod 4$. Let $\bar{\sigma} \in C$ be an element of order 4. Then, because $< \phi >$ has odd order, $\bar{\sigma}$ has a lift of order four $\sigma \in N(< \phi >)$, and by assumption $\bar{\sigma}(e_0)$=$\sigma(e_0) \in \{e_0, e_{p-1}, f_0, f_{p-k}\}$. If $\sigma(e_0) = e_0$ we get a contradiction because σ has order 4. If $\sigma(e_0) = e_{p-1}$, then, with θ_T denoting the dihedral involution given by

$$\theta_T(v_i) = v_{-i}, \quad \theta_T(e_i) = e_{-i-1}, \quad \theta_T(f_i) = f_{-i-k}\,,$$

we obtain $\theta_T \circ \sigma(e_0) = e_0$ so $\bar{\theta}_T \circ \bar{\sigma} = \mathrm{Id}$, because we know C does not contain an element of order 3. But then $\bar{\sigma}$ has order two, a contradiction. As a consequence $\bar{\sigma}(e_0) \in \{f_0, f_{p-k}\}$, and again composing with θ_T if necessary, we may assume that $\sigma(e_0) = f_0$ so that $\sigma(v_1) = v_k$. But σ normalizes ϕ, therefore

$$\sigma \circ \phi \circ \sigma^{-1} = \phi^l$$

for some l, and we see that

$$\begin{aligned}\sigma \circ \phi \circ \sigma^{-1}(v_0) &= \sigma(v_1) = v_k = \phi^k(v_0)\\ &= \phi^l(v_0) = v_l,\end{aligned}$$

from which we conclude that $k \equiv l \mod p$. As σ has order 4, we infer

$$\sigma^2 \circ \phi \circ \sigma^{-2} = \phi^{-1} = \phi^{k^2}\,,$$

thus $k^2 \equiv -1 \mod p$. The proof of the proposition (case $k \neq 0$) is completed by observing that, conversely, if $k^2 \equiv -1 \mod p$, on the one hand $p \equiv 1 \mod 4$, and on the other hand $N(< \phi >)$ contains indeed an element of order 4, namely σ defined by

$$\sigma(v_i) = v_{ki}, \quad \sigma(e_i) = f_{ki}, \quad \sigma(f_i) = e_{ki-1}.$$

Finally, the case $k = 0$ is similar but easier. One needs only check that $Aut(T_0)$ does not contain a subgroup isomorphic to M_{4p}; we leave the details to the reader.

Recall that all collapsing maps in the component $\mathcal{D}$ go from P graphs to T graphs and from S graphs to T graphs. The p-primary cohomology of all the P and S graphs is controlled by the dihedral subgroup D_{2p} and the collapsing maps are, as one checks, D_{2p}-equivariant. In case $p < 17$ where $\mathcal{D}$ is contractible, the only contribution to $H^*(Out(F_{p+1}); \mathbb{Z}_{(p)})$ coming from $\mathcal{D}$ is therefore a copy of $H^*(D_{2p}; \mathbb{Z}_{(p)})$ or $H^*(M_{4p}; \mathbb{Z}_{(p)})$, depending on whether $p \equiv 3 \mod 4$ or not. As we have already analyzed the contribution of the components $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, we get the following.

Theorem 4.11. *Let $5 \leq p < 17$ be prime. Then*

(1) *if $p \equiv 3 \mod 4$ and $* > 2p - 1$*

$$H^*(Out(F_{p+1}); \mathbb{Z}_{(p)}) = 3H^*(\Sigma_p; \mathbb{Z}_{(p)}) \oplus H^*(D_{2p}; \mathbb{Z}_{(p)})$$

(2) *if $p \equiv 1 \mod 4$ and $* > 2p - 1$*

$$H^*(Out(F_{p+1})\mathbb{Z}_{(p)}) = 3H^*(\Sigma_p; \mathbb{Z}_{(p)}) \oplus H^*(M_{4p}; \mathbb{Z}_{(p)}).$$

In the first case, $p = 5$, one obtains thus, in the range $* > 2p - 1 = 9$ the following:

$$H^*(Out(F_6); \mathbb{Z}_{(5)}) = \begin{cases} \mathbb{Z}/5, & \text{if } * \equiv 4 \mod 8 \\ 0, & \text{if } * \not\equiv 0, 4 \mod 8 \\ (\mathbb{Z}/5)^4, & \text{if } * \equiv 0 \mod 8. \end{cases}$$

For the cases $p \geq 17$ the computation gets more involved due to the fact that $\mathcal{D}$ is not contractible. For the case of $p \equiv 3 \mod 4$, all simplex stabilizers of $\mathcal{D}$ have their $\mod p$ cohomology controlled by a dihedral subgroup $D_{2p} \subset stab$. Indeed, one can choose

$$D_{2p} \subset stab(S_k)\,, \qquad D_{2p} \subset stab(T_k)\,, \qquad D_{2p} \subset stab(P_k)$$

in a way that the relevant collapsing maps are D_{2p}-equivariant. As a result, the computation in the Brown spectral sequence related to the $\mathcal{D}$ component of $X_{p+1}^{sing}/Out(F_{p+1})$ reduces to computing the cohomology of $\mathcal{D}$ with constant coefficients $H^*(D_{2p}; \mathbb{Z}_{(p)})$. The contribution to cohomology of $Out(F_{p+1})$ coming from $\mathcal{D}$ is thus in degree $l > vcd(Out(F_{p+1})$ given by

$$H^l(D_{2p}; \mathbb{Z}_{(p)}) \oplus (1 - \chi(\mathcal{D}))H^{l-1}(D_{2p}; \mathbb{Z}_{(p)})$$

(the cohomology of $BD_{2p}\times$ (bouquet of $1 - \chi(\mathcal{D})$ circles)). By taking into account the contributions of the components $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, which in degree l are just

$$3H^l(\Sigma_p; \mathbb{Z}_{(p)})$$

one obtains part (1) of Theorem 3 of the introduction.

The case $p \equiv 1 \mod 4$ is very similar. The only difference in the final formula is that the first copy of $H^*(D_{2p}; \mathbb{Z}_{(p)})$ gets replaced by $H^*(M_{4p}; \mathbb{Z}_{(p)})$, which yields part (2) of Theorem 3.

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