THE p-PRIMARY FARRELL COHOMOLOGY OF OUT($\mathbf{F}_{p-1}$)

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Abstract. We compute for an odd prime p the p-primary part of the Farrell cohomology of $Out(F_{p-1})$, the group of outer automorphisms of a free group F_{p-1} of rank $p-1$. We also determine the p-period, Yagita invariant and cohomological Krull dimension for automorphism groups of free groups.

Introduction

Let F_n denote a free group of rank n, $Aut(F_n)$ its automorphism group and $Out(F_n)$ the associated group of outer automorphisms. Recall that these groups are of finite virtual cohomological dimension. The main purpose of this note is to compute, for p a prime, the p-primary cohomology of $Aut(F_{p-1})$ and $Out(F_{p-1})$ above the virtual cohomological dimension. It is convenient to express our result in terms of Farrell cohomology. We shall write $\hat{H}^*(\Gamma;\mathbb{Z})_p$ for the p-primary part of the Farrell cohomology of a group Γ of finite *vcd*.

Theorem. *Let p be an odd prime. Then*

$$\hat{H}^*(Aut(F_{p-1});\mathbb{Z})_p \cong \hat{H}^*(Out(F_{p-1});\mathbb{Z})_p \cong \mathbb{F}_p[w,w^{-1}],$$

with w of degree $2(p-1)$.

Note that the mod p Farrell cohomology of these groups is periodic, implying that the Krull dimension of the corresponding mod p cohomology rings equals one. In Section 1 we will determine for general n the Krull dimension of the mod p cohomology rings of $Aut(F_n)$ and $Out(F_n)$; the result can be read off from the work of Smillie and Vogtmann [SV2]. Section 2 is devoted to computing the Yagita invariant of these groups; it is interesting to compare the result with the corresponding one for the case of mapping class groups (cf. [GMX]). In the third Section we present the proof of the theorem stated above; the interested reader should compare this result with the analogous computation for the mapping class group $\Gamma_{(p-1)/2}$ (cf. [X]) and for $Gl_{p-1}(\mathbb{Z})$ (cf. [A], [BE]).

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Section 1: The Krull dimension

Recall that for a group Γ of finite *vcd* the Krull dimension $\kappa_p(\Gamma)$ of the mod p cohomology of Γ equals the maximal rank of an elementary Abelian p-subgroup of Γ. Thus, if $\Lambda \to \Gamma$ is a map of groups of finite *vcd* with p-torsionfree kernel, one has

$$\kappa_p(\Lambda) \leq \kappa_p(\Gamma).$$

The natural map $Out(F_n) \to Gl_n(\mathbb{Z})$ has torsionfree kernel and thus

$$\kappa_p(Out(F_n)) \leq \kappa_p(Gl_n(\mathbb{Z})).$$

It is well-known that the rank of a maximal elementary Abelian p-subgroup of $Gl_n(\mathbb{Z})$ is $[\frac{n}{p-1}]$, where $[x]$ stands for the integral part of the real number x (cf. Minkowski [M]).

The following theorem then follows from [SV2].

Theorem 1.1. *For every prime p and $n \geq 1$ one has*

$$\kappa_p(Aut(F_n)) = \kappa_p(Out(F_n)) = [\frac{n}{p-1}].$$

Proof. Because the kernel of the natural map $Aut(F_n) \to Out(F_n)$ is torsionfree and in view of our discussion above, we have

$$\kappa_p(Aut(F_n)) \leq \kappa_p(Out(F_n)) \leq \kappa_p(Gl_n(\mathbb{Z})) = [\frac{n}{p-1}].$$

It suffices therefore to show that

$$(\mathbb{Z}/p)^m \subset Aut(F_{m(p-1)}).$$

For this, we consider the graph $G(p)$ with 2 vertices, connected by p edges, so that $\pi_1 G(p) \cong F_{p-1}$. Let $\mathbb{Z}/p$ act on $G(p)$ by permuting the edges and fixing the vertices. This action extends in an obvious way to an action of $(\mathbb{Z}/p)^m$ on $\vee_m G(p)$, yielding an injection

$$(\mathbb{Z}/p)^m \to Aut(\pi_1(\underset{m}{\vee} G(p))) \cong Aut(F_{m(p-1)}),$$

and the result follows.

Remark. The whole symmetric group Σ_p acts by automorphism on the graph $G(p)$, fixing the two vertices. Furthermore, the symmetric group Σ_m acts on $\vee_m G(p)$ by permuting the copies of $G(p)$ and fixing the basepoint. One concludes that the wreath product $\Sigma_p \int \Sigma_m := (\Sigma_p \times \cdots \times \Sigma_p) \rtimes \Sigma_m$ acts on $\vee_m G(p)$ fixing the basepoint so that

$$\Sigma_p \int \Sigma_m \subset Aut(F_{m(p-1)}).$$

Of course, m and p can here be arbitrary integers ≥ 1.

Section 2: The Yagita invariant

Recall (see [Y]) that for a group Γ of finite *vcd* and prime p the Yagita invariant $p(\Gamma)$ is defined as the least common multiple of numbers $2m(\pi)$, where $m(\pi)$ is the largest number such that

$$H^*(\Gamma; \mathbb{Z}) \xrightarrow{\text{res}} \text{im}\,(res) \subset \mathbb{Z}/p[u^{m(\pi)}] \subset H^*(\pi; \mathbb{Z}/p)$$

where $\pi \subset \Gamma$ is a subgroup of order p and $u \in H^2(\pi;\mathbb{Z}/p)$ is a generator. (If no such $\pi \subset \Gamma$ exists, one puts $p(\Gamma) = 1$). It is obvious from the definition that for any map $\Lambda \to \Gamma$ of groups of finite *vcd* with p-torsionfree kernel one has

$$p(\Lambda) \mid p(\Gamma).$$

We also recall the well-known and elementary fact that in case $\kappa_p(\Gamma) = 1$, the Yagita invariant $p(\Gamma)$ agrees with the p-period, that is, it is equal to the smallest positive integer k such that

$$H^i(\Gamma;\mathbb{Z})_p \cong H^{i+k}(\Gamma;\mathbb{Z})_p\,, \qquad \forall i > vcd(\Gamma).$$

We call such a group p-periodic; it has periodic Farrell cohomology of period $p(\Gamma)$.

Definition. *For $x \in \mathbb{R}^{>0}$ and p a prime we shall write $[x]_p$ for the largest power of p less than or equal to x; thus $x = [x]_p + y$ with $[x]_p = p^m$ and $0 \le y < (p-1)p^m$.*

Theorem 2.1. *Let p be a prime and $n \ge 1$. Then*

$$p(Aut(F_n)) = p(Out(F_n)) = 2(p-1)[\frac{n}{p-1}]_p.$$

Proof. The natural maps $Aut(F_n) \to Out(F_n) \to Gl_n(\mathbb{Z})$ with torsionfree kernels show that

$$p(Aut(F_n)) \le p(Out(F_n)) \le p(Gl_n(\mathbb{Z})).$$

From [GMX, Lemma 1.2] we infer that $p(Gl_n(\mathbb{Z}))$ divides $2(p-1)[\frac{n}{p-1}]_p$ (see also [GLT] for more information concerning the Yagita invariant of a general linear group). It suffices therefore to find a finite subgroup $\sigma \subset Aut(F_n)$ with $p(\sigma) \ge 2(p-1)[\frac{n}{p-1}]_p$. Let $p^m = [\frac{n}{p-1}]_p$. Note that $(p-1)p^m \le n$ and

$$\Sigma_p \textstyle\int \Sigma_{p^m} \subset Aut(F_{(p-1)p^m}) \subset Aut(F_n)$$

according to our remark at the end of section 1. We claim that $p(\Sigma_p \int \Sigma_{p^m})$ is larger or equal to $2(p-1)p^m$. Indeed, the normalizer of $\mathbb{Z}/p \subset \Sigma_p$ contains the holomorph $Hol(\mathbb{Z}/p)$ of $\mathbb{Z}/p$ (i.e., the split extension of $\mathbb{Z}/p$ by its automorphism group $Aut(\mathbb{Z}/p) \cong \mathbb{Z}/(p-1)$. Clearly,

$$H^*(Hol(\mathbb{Z}/p);\mathbb{Z}) = \mathbb{Z}[w]/pw \quad \text{with} \quad |w| = 2(p-1).$$

Now consider

$$Hol(\mathbb{Z}/p) \textstyle\int \Sigma_{p^m} \subset \Sigma_p \int \Sigma_{p^m}.$$

The diagonal embedding

$$\Delta : \mathbb{Z}/p \to Hol(\mathbb{Z}/p) \times \cdots \times Hol(\mathbb{Z}/p) \subset Hol(\mathbb{Z}/p) \textstyle\int \Sigma_{p^m}.$$

induces a restriction map

$$\Delta^* : H^*(Hol(\mathbb{Z}/p) \textstyle\int \Sigma_{p^m};\mathbb{Z}) \to H^*(Hol(\mathbb{Z}/p);\mathbb{Z}) \subset H^*(\mathbb{Z}/p;\mathbb{Z})$$

mapping into $\mathbb{Z}[w^{p^m}]/pw^{p^m}$. This can be seen by looking at the diagonal restriction map

$$\rho^* : H^*(Hol(\mathbb{Z}/p) \times \cdots \times Hol(\mathbb{Z}/p); \mathbb{Z}) \to H^*(Hol(\mathbb{Z}/p); \mathbb{Z}).$$

Because elements of odd degree are mapped to zero, ρ^* factors through

$$H^*(Hol(\mathbb{Z}/p); \mathbb{Z}) \otimes \cdots \otimes H^*(Hol(\mathbb{Z}/p); \mathbb{Z}) = \mathbb{Z}[w_1, \cdots w_{p^m}]/p(w_1, \cdots, w_{p^m})$$

with each w_i of degree $2(p-1)$ and $(w_1, \cdots, w_{p^m})$ denoting the ideal generated by these elements. But all elementary symmetric functions in the variables $w_1, \cdots, w_{p^m}$ map via ρ^* to 0 so that the image of Δ^* is contained in the subalgebra generated by the image of Πw_i, which is w^{p^m}. It follows that

$$p(\Sigma_p \int \Sigma_{p^m}) \geq 2(p-1)p^m,$$

completing the proof.

Remark. Note that our proof shows that

$$p(\Sigma_p \int \Sigma_{p^m}) = 2(p-1)p^m$$

and

$$p(Gl_n(\mathbb{Z})) = 2(p-1)[\frac{n}{p-1}]_p.$$

As already mentioned, the Yagita invariant coincides with the p-period in the p-periodic case, and a group of finite virtual cohomological dimension is p-periodic if and only if its elementary Abelian p-subgroups are cyclic (cf. Brown's book [B]).

Corollary 2.2. *Let p be an arbitrary prime and $n \geq 1$. Then the following holds:*

(i) *$Aut(F_n)$ is p-periodic if and only if $Out(F_n)$ is.*
(ii) *$Aut(F_n)$ and $Out(F_n)$ are p-periodic if and only if $p-1 \leq n < 2(p-1)$.*
(iii) *If $Aut(F_n)$ and $Out(F_n)$ are p-periodic, their p-period equals $2(p-1)$.*

Section 3: The p-primary cohomology

The complexity of the p-primary part of the cohomology of $Aut(F_n)$ and $Out(F_n)$ is simplest in the p-periodic case. We will completely determine the Farrell cohomology in the first case, that is, in case $n = p-1$. The next two cases, $n = p$ and $n = p+1$ respectively, will be considered in a forthcoming paper (cf. [GM]). Since the Krull dimension of $Aut(F_{p-1})$, $\kappa_p(Aut(F_{p-1}))$, equals one, the p-part of the Farrell cohomology of $Aut(F_{p-1})$ is given by (cf. Brown's book [B])

$$\hat{H}^*(Aut(F_{p-1}); \mathbb{Z})_p = \prod_{C \in K} \hat{H}^*(N_A(C); \mathbb{Z})_p$$

where K denotes a set of representatives of conjugacy classes of subgroups C of order p of $Aut(F_{p-1})$, and $N_A(C)$ the normalizer of $C \subset Aut(F_{p-1})$. Elements of order p in the automorphism group of a free group have been classified in [DS]. In particular, if $\alpha \in Aut(F_{p-1})$ has order p, then according to [DS] there exists a basis $(x_1, \cdots, x_{p-1})$ of F_{p-1} such that

$$\alpha(x_i) = \begin{cases} x_{i+1}, & 1 \leq i < p-1 \\ (x_1 x_2 \cdots x_{p-1})^{-1}, & i = p-1. \end{cases}$$

Theorem 3.1. *Let p be an odd prime. Then all elements of order p in $Aut(F_{p-1})$ are conjugate. Moreover, if $\alpha \in Aut(F_{p-1})$ has order p then $N_A(<\alpha>)$, the normalizer of the subgroup $<\alpha>$ generated by α in $Aut(F_{p-1})$, injects into $N_O(<\bar{\alpha}>)$, the normalizer of the subgroup $<\bar{\alpha}>$ generated by the image $\bar{\alpha}$ of α in $Out(F_{p-1})$.*

Proof. That the elements of order p in $Aut(F_{p-1})$ are conjugate follows from the classification theorem cited above. If ϕ lies in the kernel of

$$N_A(<\alpha>) \to N_O(<\bar{\alpha}>)$$

then ϕ is an inner automorphism of F_{p-1}, say

$$\phi(x) = txt^{-1}, t \in F_{p-1}.$$

But, assuming that α has order p, ϕ^{p-1} will centralize α so that $\phi^{p-1} \circ \alpha = \alpha \circ \phi^{p-1}$ and therefore $t^{p-1}\alpha(x)t^{1-p} = \alpha(t^{p-1})\alpha(x)\alpha(t^{1-p})$ for all $x \in F_{p-1}$. This shows that $\alpha(t^{p-1}) = t^{p-1}$, because the center of F_{p-1} is trivial for $p > 2$. Hence $t^{p-1} \in F_{p-1}^{<\alpha>}$, the group fixed by α, which is known to be a trivial group ([DS]). As a consequence, $t = e \in F_{p-1}$, since F_{p-1} is torsionfree, and we conclude that $N_A(<\alpha>) \to N_O(<\bar{\alpha}>)$ is injective.

Remark. The Theorem is false for $p = 2$ since then $Aut(F_2) = N_A(<\alpha>) \cong \mathbb{Z}/2$, but $Out(F_2) = N_O(<\bar{\alpha}>) = \{e\}$.

To establish the finiteness of the normalizers $N_A(<\alpha>)$ for α as in the Theorem 3.1, it suffices therefore to consider the normalizer $N_O(<\alpha>) \subset Out(F_{p-1})$. For this we consider the spine $\mathcal{K}_n$ of the space introduced by Culler and Vogtmann ([CV]), which was also used in [SV1]. It is a contractible $(2n-3)$-dimensional simplicial complex with vertices certain equivalence classes of *marked admissible graphs*, and $Out(F_n)$ acts on $\mathcal{K}_n$ simplicially with finite stabilizers, such that if $\gamma \in Out(F_n)$ fixes a point $x \in \mathcal{K}_n$, then it fixes the carrier of x pointwise. If $C \subset Out(F_n)$ is any subgroup, the normalizer $N_O(C)$ acts on the fixed-point space $\mathcal{K}_n^C$ and thus $\mathcal{K}_n^C = \{pt\}$ implies that $N_O(C)$ is a finite group.

Theorem 3.2. *Let p be an odd prime and $C \subset Out(F_{p-1})$ a subgroup of order p. Then $N_O(C)$, the normalizer of C in $Out(F_{p-1})$, is a finite group.*

Proof. The fixed point space $\mathcal{K}_n^G$ is known to be contractible for any finite subgroup G of $Out(F_n)$, see Krstić and Vogtmann [KV], or White [W]. In particular, $\mathcal{K}_{p-1}^C$ is non-empty and connected. We show that $\mathcal{K}_{p-1}^C = \{pt\}$ by showing that $\mathcal{K}_{p-1}^C$ does not contain any 1-simplex of $\mathcal{K}_{p-1}$. A vertex of $\mathcal{K}_{p-1}$ is represented by a pair (g, G), G an admissible graph of rank $p-1$ and $g : R \to G$ a homotopy equivalence, where R is a *rose* (i.e., a graph with a single vertex; see [SV1] for the notion and discussion of admissible graphs). The admissible graph G associated with a vertex of $\mathcal{K}_{p-1}$ is unique up to homeomorphism and is called the *graph type* of the vertex; the stabilizer of the vertex is isomorphic to the automorphism group of the graph G (which is the same as the group of components of the group of homeomorphisms of G, since G is admissible). According to [SV1], there is a unique graph of rank $p-1$ admitting an automorphism of order p (namely, the graph $G(p)$ considered in the previous section). On the other hand, the two vertices of a 1-simplex of $\mathcal{K}_{p-1}$ have always distinct graph type (one is obtained from the other by a *forest collapse*). Thus $\mathcal{K}_{p-1}^C$ contains no 1-simplex, which proves that $N_O(C)$ is finite.

We can now prove the first half of the Theorem of the introduction.

Theorem 3.3. *Let p be an odd prime. Then*

$$\hat{H}^*(Aut(F_{p-1});\mathbb{Z})_p \cong \mathbb{F}_p[w, w^{-1}],$$

with w of degree $2(p-1)$.

Proof. We know from our earlier discussion that $Aut(F_{p-1})$ contains elements of order p. By Theorem 3.1 we conclude that there is a unique conjugacy class of subgroups of order p in $Aut(F_{p-1})$ and, if C is such a subgroup, the natural map

$$N_A(C) \to Aut(C) \cong \mathbb{Z}/p-1$$

is surjective. Since $N_A(C)$ is contained in the finite group $N_O(C)$, it injects via the natural map $Aut(F_{p-1}) \to Gl_{p-1}(\mathbb{Z})$ into $Gl_{p-1}(\mathbb{Z})$, which by Minkowski's Theorem [M] does not contain any p-subgroups of order larger than p. It follows that the p-Sylow subgroup of the (finite) group $N_A(C)$ is cyclic of order p. Hence, $N_A(C)$ is a split extension of C by a finite group of order prime to p and one has therefore a natural surjection onto the holomorph of C,

$$N_A(C) \to C \rtimes Aut(C) =: Hol(C)$$

with finite kernel Q of order prime to p. In particular, this projection will induce an isomorphism on the p-primary part of cohomology. Thus

$$H^*(N_A(C);\mathbb{Z})_p \cong H^*(Hol(C);\mathbb{Z})_p \cong \mathbb{Z}[w]/pw$$

with w of degree $2(p-1)$. Moreover, we infer that

$$\hat{H}^*(Aut(F_{p-1});\mathbb{Z})_p \cong \hat{H}^*(N_A(C);\mathbb{Z})_p \cong \mathbb{F}_p[w, w^{-1}]$$

with w of degree $2(p-1)$.

To obtain the corresponding result for $Out(F_{p-1})$, we first need to verify that $Out(F_{p-1})$ contains just one conjugacy class of subgroups of order p. This follows from the following Lemma.

Lemma 3.4. *Let p be an odd prime. Every subgroup $C \subset Out(F_{p-1})$ of order p lifts to a subgroup $\tilde{C} \subset Aut(F_{p-1})$ of order p.*

Proof. We make use of Zimmermann's Realization Theorem [Z], (see also Culler [C]), which implies the following. Given $C \subset Out(F_{p-1})$ of order p, there exists a finite, connected graph H containing a subgroup $D \subset Aut(H)$ of order p such that D maps onto C via an isomorphism $\pi_1(H) \to F_{p-1}$. To see that C lifts to a subgroup $\tilde{C} \subset Aut(F_{p-1})$ of order p, it suffices therefore to check that the D-action on H has a fixed-point. For this, let $f \in D$ denote a generator and consider

$$f_* : H_1(H;\mathbb{Q}) \to H_1(H;\mathbb{Q}).$$

Since f maps to an element of order p in $Gl_{p-1}(\mathbb{Q})$ via the maps

$$Aut(H) \to Out(F_{p-1}) \to Gl_{p-1}(\mathbb{Z}) \to Gl_{p-1}(\mathbb{Q}),$$

we conclude that the trace

$$\mathrm{Tr}(f_*) = -1.$$

We use here the well-known fact that every matrix in $Gl_{p-1}(\mathbb{Q})$ of order p has trace -1. It follows that the Lefschetz number of f satisfies

$$\Lambda(f) = 1 - (-1) = 2,$$

and we conclude that $f : H \to H$ has a fixed-point.

Remark. The result that $Out(F_{p-1})$ contains only one conjugacy class of subgroups of order p follows also immediately from the fact that there is only one admissible graph of rank $p-1$ on which the cyclic group of order p can act ([SV2]), together with the Culler–Zimmermann realization theorem (loc. cit.).

Theorem 3.5. *Let p be an odd prime. Then*

$$\hat{H}^*(Out(F_{p-1});\mathbb{Z})_p \cong \mathbb{F}_p[w, w^{-1}]$$

with w of degree $2(p-1)$.

Indeed, $Out(F_{p-1})$ contains a unique conjugacy class of subgroups of order p and the normalizer $N_O(C)$ of such a subgroup is a finite group, mapping onto $Hol(C)$ with kernel of order prime to p. The rest of the argument is then as in the proof for Theorem 3.3. Note, because $vcd(Out(F_n)) = 2n - 3$ (cf. [CV]), Theorem 3.5 determines also the ordinary cohomology of $Out(F_{p-1})$ in degrees $> 2p - 5$.

References

[A] A. Ash, *Farrell Cohomology of $GL(n,\mathbb{Z})$*, Israel. J. Math. **67** (1989), 327–336.

[B] K.S. Brown, *Cohomology of groups*, Springer–Verlag New York, Heidelberg, Berlin, 1982.

[BE] B. Bürgisser and B. Eckmann, *The p-torsion of the Farrell-Tate cohomology of $GL(n, O_S(K))$ and $SL(n, O_S(K))$*, Mathematika **31** (1984), 89–97.

[C] M. Culler, *Finite groups of outer automorphisms of a free group*, Contemporary Mathematics **33** (1984), 197–207.

[CV] M. Culler and K. Vogtmann, *Moduli of automorphisms of free groups*, Invent. math. **84** (1986), 91–119.

[DS] J. Dyer and G.P. Scott, *Periodic automorphisms of free groups*, Communications in Algebra **3** (1975), 195–201.

[GLT] H.H. Glover, I. Leary and C.B. Thomas, *The Yagita invariant of general linear groups*, (to appear in: Proceedings of the Barcelona conference on algebraic topology, 1994).

[GM] H.H. Glover and G. Mislin, *The p-primary cohomology of $Out(F_n)$ in the p-rank one case*, (in preparation).

[GMX] H.H. Glover, G. Mislin and Y. Xia, *On the Yagita invariant of mapping class groups*, Topology **33** (1994), 557–574.

[KV] S. Krstić and K. Vogtmann, *Equivariant outer space and automorphisms of free-by-finite groups*, Comment. Math. Helv. **68** (1993), 216–262.

[M] H. Minkowski, *Zur Theorie der positiven quadratischen Formen*, Crelles J. **101** (1887), 196–202.

[SV1] J. Smillie and K. Vogtmann, *A generating function for the Euler characteristic of* $Out(F_n)$, J. Pure and Appl. Algebra **44** (1987), 329–348.

[SV2] J. Smillie and K. Vogtmann, *Automorphisms of graphs, p-subgroups of* $Out(F_n)$ *and the Euler characteristic of* $Out(F_n)$, J. Pure and Appl. Algebra **49** (1987), 187–200.

[W] T. White, *Fixed points of finite groups of free group automorphisms*, Proc. Amer. Math. Soc. **3** (1993), 681–688.

[X] Y. Xia, *The p-torsion of the Farrell cohomology of* $\Gamma_{(p-1)/2}$, Topology '90 (B. Apanasov, W.D. Neumann, A.W. Reid and L. Siebenmann eds.), Walter de Gruyter (1992), 391–398.

[Y] N. Yagita, *On the dimension of spheres whose pruduct admits a free action by a non-abelian group*, Quart. J. Math. Oxford **36** (1985), 117–127.

[Z] B. Zimmermann, *Über Homöomorphismen n-dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen*, Comment. Math. Helv. **56** (1981), 424–486.

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