Faithfully Flat Descent for Morphism of Schemes

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Theorem. *Let A be a ring and let B be an $A-$algebra. If B is a faithfully flat $A-$algebra, then*

$$0 \to A \xrightarrow{f} B \xrightarrow{d} B \otimes B$$

is exact, where $d(b) = 1 \otimes b - b \otimes 1$.

Proof. Let A be a ring and let B be an $A-$algebra. If the sequence in the statement split at the first position, i.e., there exist a ring homomorphism $h : B \to A$ such that $h \circ f = id$, then the sequence is exact. First note that $d(f(a)) = 1 \otimes f(a) - f(a) \otimes 1 = 0$ as the tensor product is over A. Here we consider another map $h_1 = f \circ h + h_2 \circ d$, where $h_2 = h \otimes id$. Then $h_1(b) = f \circ h(b) + h_2 \circ d(b) = f(h(b)) + h_2(1 \otimes b - b \otimes 1) = f(h(b)) + b - f(h(b)) = b$. Thus $h_1 = id_B$. Then $d(b) = 0 \Rightarrow h_2(d(b)) = 0 \Leftrightarrow f(h(b)) = b$. Thus the sequence exact.

Now we assume that B is a faithfully flat $A-$algebra. Then we tensor the sequence with B, we get $0 \to B \to B \otimes B \to B \otimes B \otimes B$, as it is obvious admits a splitting by $a \otimes b \mapsto ab$, the sequence is exact. As B is faithfully flat, the original sequence is exact. □

Theorem. *With same assumption in last theorem. Then*

$$0 \to A \to B \xrightarrow{d_0} B \otimes_A B \to ... \xrightarrow{d_{r-2}} B^{\otimes r}$$

is exact, where $d_{r-1}(b_0 \otimes ... \otimes b_{r-1}) = \sum_i (-1)^i b_0 \otimes ... b_{i-1} \otimes 1 \otimes b_i \otimes ... b_{r-1}$.

Proof. This proof is similar to the last one. We can reduce to the case where it admits a splitting. Let g be the splitting. Consider $h_r(b_0 \otimes ... b_{r+1}) = g(b_0)(b_1 \otimes ... b_{r+1})$. Then $h_{r+1} \circ d_{r+1} + d_r \circ h_r = id$. Then we just run through last proof. □

Corollary. *Let $\pi : U \to X$ be a morphism of affine schemes, and let pr_1 and pr_2 be the projection of $U \times_X U \to U$. If π is flat and surjective, then for any affine scheme T, we have*

$$Hom(X, T) \xrightarrow{\circ \pi} \Pi_i Hom(U_i, T) \underset{\circ pr_2}{\overset{\circ pr_1}{\rightrightarrows}} \Pi_{i,j} Hom(U_i \times_T U_j, T).$$

Proof. This is just the corresponding diagram of the first theorem in the affine scheme language. □

Definition. *Let T be a scheme. A fpqc covering is a family of morphisms $\{\phi_i : T_i \to T\}_{i \in I}$ for some index I, such that (1) each ϕ_i is flat, (2) for each affine open subspace $U \subset X$, there exist a finite index set K, and a map $j : K \to I$, and affine open subspace $U_{j(k)} \subset T_{j(k)}$ for all k such that $U = \cup_k \phi_{j(k)}(U_{j(k)})$.*

Example. *Any Zariski open cover is a fpqc covering.*

Theorem. *A fppf covering is a fpqc covering.*

Definition. *Let S be a scheme and let Sch/S denote the category of schemes over S. Consider a presheaf of sets, a contravariant functor $F : Sch/S \to Set$. We say F is a sheaf in the fpqc topology if for every fpqc covering $\phi_i : T_i \to T$, we have a equializer diagram:*

$$F(T) \longrightarrow \Pi_i F(U_i) \rightrightarrows \Pi_{i,j} F(U_i \times_T U_j).$$

Theorem. *Let F be a presheaf on Sch/S. F is a sheaf if and only if (1) It is a sheaf for Zariski topology (2) for every faithfully flat morphism $Spec(B) \to Spec(A)$ affine scheme over S, we have an equilizer:*

$$F(Spec(A)) \longrightarrow F(Spec(B)) \rightrightarrows F(Spec(B \otimes_A B)).$$

Proof. It is easy to see only if part. Assume we have (1) and (2). Let $\{f_i : T_i \to T\}$ be a fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that their image in $F(T_i \times_T T_j)$ are same. Then we want to show there exist a unique $s \in F(T)$ such that $s|_{T_i} = s_i$. Let $W \subset T$ be the maximal open subset with the property that there is a unique $s \in F(W)$ such that $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$. Such a set exists as we require it to satisfies the sheaf condition for Zariski topology. Now we show that $W = T$.

Let $t \in T$, let U be an open affine neighbourhood of t. Then we can find an affine covering $\{U_i \to U\}_{i \in J}$, which is fpqc, and is a refinement of $\{T_i \times_T U \to U\}_{i \in I}$ and J is a finite set, by maps $h_j : U_j \to T_{i_j} = U_j \times_T T_i$. Then we have an element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$, by property (2). Now For any scheme $V \to U$, there is a unique section $s_V \in F(V)$ such that $F(h_j \circ pr)(s_{i_j}) = s_V|_{V \times_U U_j}$, where pr is the projection to U_j. This is ture for affine case by property (2), and is ture for general case by property (1). Then we have $s|_V = s_V$. Now we consider $V = U \times_T T_i$, we have $s_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$. □

Corollary. *For each scheme Y over S the presheaf $T \mapsto Hom(T, Y)$ is a sheaf in fpqc topology.*

Proof. This is by the last corollary the the last theorem. □

Example. *Let X be a scheme, then the followings are sheaves: (1) $G_a(T) = \Gamma(T, O_T)$; (2) $G_m(T) = \Gamma(T, O_T)^\times$; (3) For ever $n \in \mathbb{Z}$, and positive, $\mu_n(T) = \{x \in G_m(T) | x^n = 1\}$.*

Proof. (1) This is the representable scheme of $Spec(\mathbb{Z}[t])$. (2) This is the representable scheme of $Spec(\mathbb{Z}[t, t^{-1}])$. (3) This is the representable scheme of $Spec(\mathbb{Z}[t]/(t^n - 1))$. □