Descent for Quasi-coherent Sheaves

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November 2018

We mainly follow the part of descent in stack project([1])

1 Descent Datum for Quasi-coherent Sheaves

The general descent datum for sheaves is a generalization for that of gluing sheaves which are given on open covering of a scheme.

Definition 1.1. Let S be a scheme. Let $\{f_i : S_i \to S\}_{i\in I}$be a family of morphisms with target S

(1) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to $\{f_i : S_i \to S\}_{i\in I}$ is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for any $i \in I$, and for every $i, j \in I$, $\varphi_{ij} : pr_0^*\mathcal{F}_i \to pr_1^*\mathcal{F}_j$ are isomorphisms for quasi-coherent sheaves on $S_i \times_S S_j$, and these isomorphisms satisfy cocycle conditions:

$$\begin{tikzcd}
pr_0^*\mathcal{F}_i \arrow[rr, "pr_{02}^*\varphi_{ik}"'] \arrow[dr, "pr_{01}^*\varphi_{ij}"'] & & pr_2^*\mathcal{F}_k \\
& pr_1^*\mathcal{F}_j \arrow[ur, "pr_{12}^*\varphi_{jk}"'] &
\end{tikzcd}$$

These are $\mathcal{O}_{S_i\times_S S_j\times_S S_k}$-mod isomorphisms, for every $i, j, k \in I$.

(2) Moreover, we can define morphism of descent datum $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$, which is to give a family of morphisms of sheves of module, $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$, which are compatible with the "gluing" morphisms, i.e the following commutative diagram:

$$\begin{tikzcd}
pr_0^*\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] \arrow[d, "pr_0^*\psi_i"'] & pr_1^*\mathcal{F}_j \arrow[d, "pr_1^*\psi_j"] \\
pr_0^*\mathcal{F}'_i \arrow[r, "\varphi'_{ij}"] & pr_1^*\mathcal{F}'_j
\end{tikzcd}$$

Note. *(1) about the notation: for a finite product of schemes, denote pr_i, as projection to the ith(from the left) component, i begins from 0. While pr_{ij} denote the projection to the product of i,j component.*

(2) In the diagram in (1) we used the natural isomorphism from pr_i^* to $pr_{ij}^* \circ pr_0^*$ to identify the two functors, in the discussion below we always use these identifications, and since it is isomorphism of functors and induced from being left adjoint of push forward, we will see that we can get all the natural properties(commutative diagrams) we want.

We can pull back the cocycle condition to get more sense on descent datum, like what we do in the following lemma.

Lemma 1.1. *By natural identification of $S_i \times_S S_j \cong S_j \times_S S_i$, we can get from φ_{ji} a morphism on $S_i \times_S S_j$ from $pr_1^*\mathcal{F}_j$ to $pr_0^*\mathcal{F}_i$, which is still denoted by φ_{ji}, then $\varphi_{ji} \circ \varphi_{ij} = id$*

Proof. pull back the cocycle condition on $S_i \times_S \times_S S_j \times S_i$ to $S_i \times_S S_j$ by the morphism $(a, b) \mapsto (a, b, a)$, and use the identifications for pull back functor($f^* \circ g^* \cong (g \circ f)^*$), we get that $\varphi_{ji} \circ \varphi ij = \varphi_{ii}|_{S_i\times_S S_j}$, where the restriction is along $(a, b) \mapsto (a, a)$, thus it factor through the diagonal map Δ of $S_i \to S_i \times_S S_i$, while pull back concycle condition on $S_i \times_S S_i \times_S S_i$ to S_i, also by cocyle condition we get $\Delta^*\varphi_{ii} \circ \Delta^*\varphi_{ii} = \Delta^*\varphi_{ii}$, thus $\Delta^*\varphi_{ii} = id$. ∎

Note. *To guarantee that we can get the desired commutative diagram after doing so many identifications of pull back functors(like here, once we pull back cocycle relations, we do identifications), we need to explain the natural property of these identifications:*

(1) since $(g\circ f)^* \cong f^*\circ g^*$ is natural isomorphism of functors we have

$$\begin{tikzcd}
Hom(\mathcal{F},\mathcal{G}) \arrow[rr, "(g\circ f)^*"'] \arrow[dr, "f^*\circ g^*"'] & & Hom((g\circ f)^*\mathcal{F},(g\circ f)^*\mathcal{G}) \\
& Hom(f^*\circ g^*\mathcal{F}, f^*\circ g^*\mathcal{G}) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

(2) commuative diagram of natural transformations of functors:

$$\begin{tikzcd}
(h\circ g\circ f)^* \arrow[r, "\cong"'] \arrow[d, "\cong"'] & f^*\circ(h\circ g)^* \arrow[d, "\cong"'] \\
(g\circ f)^*\circ h^* \arrow[r, "\cong"'] & f^*\circ g^*\circ h^*
\end{tikzcd}$$

This follows from the fact all the identifications are made using the property of all being left adjoint of $h_*\circ g_*\circ f_*$

We can pull back descent datum by the following lemma:

Lemma 1.2. *Giving a morphism of families of morphism with the same target, $\{S_i\to S\}\to\{T_i\to T\}$ is to give a triple $\{\psi: S\to T, \alpha: I\to J, \psi_i: S_i\to T_\alpha(i)\}$, st ψ_i compatible with ψ, then we can pull back descent datum $(\mathcal{F}_i,\varphi_{ij})$ for $\{T_i\to T\}$ to $(\psi_i^*\mathcal{F}_{\alpha(i)}, (\psi_i\times\psi_{i'})^*\varphi_{\alpha(i)\alpha(i')})$ is a descent datum for $\{S_i\to S\}$(here again we use identifications for f^* functors).Moreover the pull back , up to isomorphism only depend on the $\psi: S\to T$*

Proof. (1) the cocycle condition directly follows from that of the original descent datum so the pullback one is indeed descent datum.

(2) for a different $\alpha': I\to J$, $\psi_i': S_i\to T_{\alpha'(i)}$, the isomorphism of descent datum is given by $u_i = (\psi_i\times\psi_i')^*\varphi_{\alpha(i)\alpha'(i)}$, to see the u_i compatible withh gluing morphisms, for example pull back the cocyle condition on $T_{\alpha(i)}\times_T T_{\alpha(i')}\times_T T_{\alpha'(i')}$ to $S_i\times_S S_i'$ by obvious morphisms and see what can one get. ∎

Note. *Moreover, the pull back defines functor on category of descent datum, for a morphism of families f , we still denote the pull back of descent datum by f^*, and for composition of pull back, we still have $(g\circ f)^*\cong f^*\circ g^*$. And f_1, f_2 if the same on the morphism for target, we have natural isomorphism $f_1^*\cong f_2^*$*

Lemma 1.3. *Descent datum for $\{f_i: S_i\to S\}_{i\in I}$ is the same with that of $\{\coprod_i f_i: \coprod_i S_i\to S\}$ by pullback along the obvious morphism $\{f_i: S_i\to S\}_i\to\{\coprod_i f_i:\coprod_i S_i\to S\}$between the two families.*

Proof. The quasi-inverse functor is given by for a descent datum $(\mathcal{F}_i,\varphi_{ij})$ for $\{S_i\to S\}$ will give a sheaf $\mathcal{F}=\coprod\mathcal{F}_i$ on $T=\coprod S_i$ while $T\times_S T=\coprod_{i,j} S_i\times_S S_j$, it is clear that $\varphi=\coprod_{i,j}\varphi_{ij}$ gives the gluing morphism for $\mathcal{F}$ ∎

Definition 1.2. canonical descent datum: the trivial descent datum $\mathcal{F}, id$ pulled back by the $\{f_i: S_i\to S\}_{i\in I}\to\{S=S\}$

Definition 1.3. effective descent datum: descent datum which is isomorphic to the canonical descent datum.

Proposition 1.4. *Deduce the descent datum for descent for module: for $\{spec\mathbf{B}\to spec\mathbf{A}\}$ the descent datum is equivalent to give a (N,φ), N is a B-mod, $\varphi: N\otimes_R A\to A\otimes_R N$ is $B\otimes_A B$-mod isomophism which satisfies the cocycle condition: if $\varphi(n\otimes 1)=\sum a_i\otimes n_i$ then $\sum a_i\otimes 1\otimes n_i=\sum a_i\otimes\varphi(n_i\otimes 1)$*

The cocycle condition could be drawed like:

$$\begin{tikzcd}
N\otimes_A B\otimes_A B \arrow[rr, "\varphi_{01}"'] \arrow[dr, "\varphi_{02}"'] & & B\otimes_A N\otimes_A B \arrow[dl, "\varphi_{12}"'] \\
& B\otimes_A B\otimes_A N &
\end{tikzcd}$$

Where if $\varphi(n\otimes 1)=\sum a_i\otimes x_i$, then $\varphi_{01}=\varphi\otimes id$, $\varphi_{12}=id\otimes\varphi$,$\varphi_{02}(n\otimes 1\otimes 1)=\sum a_i\otimes 1\otimes x_i$

While the morphism of descent datum $\psi : (N, \varphi) \to (N', \varphi')$ is a B-mod isomorphism ψ such that $\varphi'(\psi(n) \otimes 1) = \sum a_i \otimes \psi(x_i)$ if $\varphi(n \otimes 1) = \sum a_i \otimes x_i$

The canonical descent datum denoted by $(B \otimes_A M, can)$, the $can : (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$ sends $(b \otimes m) \otimes b' \mapsto b \otimes (b' \otimes m)$

Note. *Corresponding to the notation in the background of scheme. We denote $\tau_i^1 : B \to B \otimes_A B$ the embeding to the ith component,$i = 0, 1$. $\tau_{ij}^2 : B \otimes_A B \to B \otimes_A B \otimes_A B$ as embeding to i, j th component, $(i, j) = (0, 1), (1, 2), (0, 2)$. For a ring morphism $f : A \to B$, denote functors: $f^*M = B \otimes_f M$, $f_*N = N$, viewed as A-mod.*

Proof. Notice that we have global section functor from category of q.c sheaves over affine schemes to categoory of modules, which commute with pushout functors, thus also commute up to natural isomorphisms witth pullback. Therefore, after we add global section functor and substitute the pullback used in the definition of descent datum of q.c sheaves to pullback in the category of module, we get descent datum for modules, which forms an equivalent category with that of q.c sheaves.

To get explicit formulas, we use identifications: $(B \otimes_A B) \otimes_{\tau_0^1} N \cong N \otimes_A B$,$(B \otimes_A B) \otimes_{\tau_1^1} N \cong B \otimes_A N$, $(B \otimes_A B \otimes_A B) \otimes_{\tau_{01}^2} (N \otimes_A B) \cong N \otimes_A B \otimes_A B$, $(B \otimes_A B \otimes_A B) \otimes_{\tau_{01}^2} (B \otimes_A N) \cong B \otimes_A N \otimes_A B$, and ${\tau_{01}^2}^* \varphi$ becomes $\varphi \otimes_A id$ under this identification, similarl calculations for other formulas. ∎

2 Descent for Modules

For a descent datum (N, φ) for $f : A \to B$, we can associate the following A-mod $r(N, \varphi)$ as the equalizer of

$$N \rightrightarrows B \otimes_A N$$

Where the upper morphism is $n \mapsto 1 \otimes n$ the lower one is $n \mapsto \varphi(n \otimes 1)$.

Lemma 2.1. *$A \to B$is faithfully flat, and when $(N, \varphi) = (B \otimes_A M, can)$, we have $r(N, \varphi) = M$, i.e the equalizer diagram:*

$$M \longrightarrow B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A N$$

where the upper morphism: $b \otimes m \mapsto 1 \otimes b \otimes m$, the lower one: $b \otimes m \mapsto b \otimes 1 \otimes m$

Proof. Notice that this is equivalent to prove the associated complex to the equalizer problem to be exact at degree 0 and 1. Then we can do faithfully flat base change $A \to A'$ then we have natural isomorphisms which form chain complex:

$$\begin{tikzcd}
0 \arrow[r] & A' \otimes_A M \arrow[r] \arrow[d, "="'] & A' \otimes_A (B \otimes_A M) \arrow[r] \arrow[d, "\cong"'] & A' \otimes_A (B \otimes_A B \otimes_A M) \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & B' \otimes_{A'} M' \arrow[r] & B' \otimes_{A'} B' \otimes_{A'} M' \arrow[r] & 0
\end{tikzcd}$$

Where the lower complex is the one associated to $(B' \otimes_{A'} M', can)$, where $B' = A' \otimes_A B$, $M' = A' \otimes_A M$. Since $A \to A'$ faithfully flat it keeps condition on being exact, thus by the diagram above it suffices to prove the lemma after doing a f.f base change, esp. we can choose $A \to B$ to do base change, then for $A' \to B'$, it has a section.

But for the case there is a section from B to A, we have proved the extened complex

$$0 \longrightarrow A \longrightarrow B \longrightarrow B \otimes_A B \longrightarrow B \otimes_A B \otimes_A B \longrightarrow ...$$

is null-homotopic in the last talk, while the complex we concern is part of the the tensor over A of it with M, thus exact in degree 0 and 1. ∎

Lemma 2.2. *(N, φ) is effective iff $B \otimes_A r(N, \varphi) \cong N$*

Proof. The necessary part follows from lemma 2.1. Conversely, we need to show $\psi : B \otimes_A r(N, \varphi) \to N$ gives morphism of descent datum, but the equation we have to check just tells $\varphi(m \otimes 1) = 1 \otimes m, m \in r(N, \varphi)$ ∎

Corollary 2.3. (N,φ) *effective iff effective after a faithfully flat base change* $A \to A'$, (N',φ') *effective, where* $N' = A' \otimes_A N$, *while* φ' *defined as follow:*

$$\begin{tikzcd}
N' \otimes_{A'} B' \arrow[r, "\varphi'"'] \arrow[d, "\cong"'] & B' \otimes'_A N' \arrow[d, "\cong"'] \\
A' \otimes_A (N \otimes_A B) \arrow[r, "id\otimes\varphi"'] & A' \otimes_A (B \otimes_A N)
\end{tikzcd}$$

(in fact by definition, we can check the (N',φ') *here is just isomorphic to the pull back of descent datum* (N,φ) *along the base change)*

Proof. It is easy to check by definition that the complex associated to (N',φ') is by tensoring A' with that of (N,φ) over A. Thus we get $r(N',\varphi') \cong A' \otimes_A r(N,\varphi)$, and moreover the map $B' \otimes_{A'} r(N',\varphi') \to N'$ is isomorphic to $A' \otimes_A (B \otimes_A r(N,\varphi)) \to A' \otimes_A N$. While a morphism being isomorphim iff isomrphism after a f.f base change.By lemma 2.2 we get our conclusion. ■

Theorem 2.4. *All the descent datum for a faithfully flat ring extension are effective.*

Proof. By the last lemma, we can prove after doing the f.f base change $A \to B$, thus we can prove under the assumption that there is a section. But the in general(even in the setting of descent of sheaves) descent datum is always effective when there is a section:

$p : S \to T$ be a morphism of schemes, it has a section $s : T \to S$ st $p \circ S = id_T$, then all the descent datum for $p : S \to T$ effective, because we following diagram:

$$\begin{tikzcd}
S \arrow[r, "p"'] \arrow[dr, "p"'] & T \arrow[r, "s"'] \arrow[d, "id"'] & S \arrow[dl, "p"] \\
& T &
\end{tikzcd}$$

The $s \circ p$ gives refinement from $p : S \to T$ to itself, thus by lemma 1.1, pullback by $p^* \circ s^*$ give category equivalence of descent datum for $p : S \to T$, show descent datum ϵ for p effective, it suffices to show for $p^* \circ s^*\epsilon$, but $s^*\epsilon$ is trivially effective. ■

3 Proof of Faithfully Descent of Quasi-coherent Sheaves

We can just consider fpqc cover with a single morphism $f : S \to T$

Lemma 3.1. *For* $\{f : X \to Y\}$ *being a fpqc covering, for quasi-coherent sheaves* $\mathcal{F}, \mathcal{G}$ *on* Y*, we have the equalizer diagram:*

$$Hom(\mathcal{F},\mathcal{G}) \longrightarrow Hom(\mathcal{F}|_X, \mathcal{G}|_X) \rightrightarrows Hom(\mathcal{F}|_{X\times_Y X}, \mathcal{G}|_{X\times_Y X})$$

For the equalizer, the upper morphism is by pullback by first projection, lower morphism is given by pullback by second projection, using natural identification for pull back functors.

Proof. First reduce to the affine case, for general Y choose affine cover U_i of Y, then the $V_i = f^{-1}(U_i)$ cover X, then we have:

$$\begin{tikzcd}
Hom(\mathcal{F},\mathcal{G}) \arrow[r] \arrow[d, hook] & Hom(\mathcal{F}|_X, \mathcal{G}_X) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & Hom(\mathcal{F}|_{X\times_Y X}, \mathcal{G}|_{X\times_Y X}) \arrow[d, hook] \\
\prod_i Hom(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}) \arrow[r] & \prod_i Hom(\mathcal{F}|_{V_i}, \mathcal{G}|_{V_i}) \arrow[r, shift left] \arrow[r, shift right] & Hom(\mathcal{F}|_{V_i\times_{U_i} V_i}, \mathcal{G}|_{V_i\times_{U_i} V_i})
\end{tikzcd}$$

Assume the lemma true for affine target, then by the equalizer diagram in the lower line, we first conclude the pull back morphism of sheaves along fpqc cover is faithful. Then by diagram chasing, for any $\phi : \mathcal{F} \to \mathcal{G}$ satisfying equalizer condition, then there is $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$ st $\psi_i|_{V_i} = \phi|_{V_i}$. We claim that these ψ_i glue to a morphism ψ on X, because they agree after pullback on $V_i \times_Y V_j$. While the later one follows from ϕ satisfying the equalizer diagram, by the following commuative diagram:

$$\begin{tikzcd} V_i \arrow[d] & V_i \times_Y V_j \arrow[l] \arrow[r] \arrow[d] & V_j \arrow[d] \\ X & X \times_T Y \arrow[l] \arrow[r] & X \end{tikzcd}$$

Thus since $V_i \times_T V_j \to U_i \cap U_j$ is also fpqc cover, thus ψ_i, ψ_j agree on $U_i \cap U_j$. Thus we get the reduction to affine target.

For the T affine, then we have a affine refinement $S' \to S$, assume the equalizer diagram true for $S' \to T$, then by diagram chasing method similar to above we can deduce the equalizer diagram for $S \to T$.

Finally for the affine case, $spec(B) \to spec(A)$, $A \to B$ faithfully flat, then using the global section functor the equalizer diagram becomes:

$$Hom_A(M,N) \longrightarrow Hom_B(B \otimes_A M, B \otimes_A N) \rightrightarrows Hom_{B \otimes_A B}(B \otimes_A B \otimes_A M, B \otimes_A B \otimes_A N)$$

One can check that the diagram follows from that of lemma 2.1 by using adjoint property of tensor product and and fact Hom is left exact.

■

Proposition 3.2. *For a fpqc cover $\{S_i \to S\}$ The functor assocaites $\mathcal{O}_S$-mod to canonical descent datum is fully faithful.*

Proof. Just observe that the morphism of $(f^*\mathcal{F}, can)$ and $(f^*\mathcal{G}, can)$ are just those in $Hom_{O_S}(f^*\mathcal{F}, f^*\mathcal{G})$ satisfying the equalizer condition, thus follows from lemma 3.1. ■

Lemma 3.3. *$S \to T$ is a fpqc cover, $S' \to T$ fpqc which refines $S \to T$, then the pull back of descent datum for $S \to T$ to $S' \to T$ would be fully faithful.*

Proof.

$$\begin{tikzcd} S \arrow[r] & T \\ S \times_T S' \arrow[u, "p_0"] \arrow[r, "p_1"'] & S' \arrow[u] \arrow[ul, "t"'] \end{tikzcd}$$

Look at the fibred diagram, then $t \circ p_1$ and p_0 gives two refinement from $S \times_T S'$ to S. Thus they induce isomorphic pullback, thus to show t^* fully faithful it suffice to show that of p_1 and p_0, thus we reduce to the case of $S' \to S$ is a fpqc cover.

Now we can use lemma 2.1, so we deduce first that the pullbcak of descent datum t^* will be faithful. Conversely, giving a $\psi' : t^*\mathcal{F} \to t^*\mathcal{G}$ as morphism of descent datum $(\mathcal{F}, \varphi)$ and $(\mathcal{G}, \varphi')$, we will show it must come from a psi, st $\psi' = t^*\psi$ by using lemma 3.1. So we must show after pullback along $S' \times_S S' \rightrightarrows S'$, $pr_0^*\psi' = pr_1^*\psi'$ after identification of pullback functors, but since ψ' is a morphism of descent datum $t^*(\mathcal{F}, \varphi)$ and $t^*(\mathcal{G}, \varphi')$, so for $S' \times_T S' \rightrightarrows S'$, we have $\varphi'|_{S' \times_T S'} \circ pr_0^*\psi' = pr_1^*\psi' \circ \varphi|_{S' \times_T S'}$

But after pullback along $S' \times_S S' \to S' \times_T S'$ we have $(\varphi|_{S' \times_T S'})|_{S' \times_S S'} = id$, the same for φ'. This is because the following (fibred) diagram, and notice that $\Delta^*\varphi = id$.

$$\begin{tikzcd} S' \times_T S' \arrow[r] & S \times_T S \\ S' \times_S S' \arrow[u] \arrow[r] & S \arrow[u, "\Delta"'] \end{tikzcd}$$

Thus pullback the compatible condition of morphism of descent datum to $S' \times_S S'$, we get that ψ' satisfies the equalizer condition. ■

Corollary 3.4. *For $f : S \to T$ when T affine, all descent datum for f is effective.*

Proof. We can find an affine refinement $S' \to S$, S' affine, so for a descent datum xi for f, by result on affine case, there is $\mathcal{F} \in Qcoh(T)$, st $\xi|_{S'} \cong (\mathcal{F}|_{S'}, can)|_{S'}$, thus by last lemma, $\xi \cong (\mathcal{F}|_S, can)$. ■

Theorem 3.5. *All descent datum for fpqc covering are effective.*

Proof. For $f : S \to T$, choose affine cover U_i of T, $V_i = f^{-1}(U_i)$, then we can refine S by $\coprod V_i$, thus by lemma 3.3, it suffice to show effective property for $\coprod V_i$. Now we have $\coprod V_i \to \coprod U_i \to T$, the descent for open cover $\coprod U_i \to T$ is always effective by gluing sheaves in usual sense. Thus it suffices to show that for a descent datum $\xi = (\coprod \mathcal{G}_i, \varphi_{ij})$ for $\coprod V_i \to T$, is pulled back by a descent datum $(\coprod \mathcal{F}_i, u_{ij})$ for $\coprod U_i \to T$.

But for $V_i \to U_i$, by lemma 3.4, we know that there is $\mathcal{F}_i$ on U_i, such that for descent datum $\xi_i = \xi|_{V_i}$ for $V_i \to U_i$ is effective and isomorphic to $(\mathcal{F}_i|_{V_i}, can)$ by ψ_i. We can pullback ξ_i, ξ_j to descent datum for $V_i \times_T V_j \to U_i \cap U_j$, and we have natural isomophism between results of these two pullback according to lemma 1.2. Thus we get correspondent morphism $\mathcal{F}_i|_{V_i \times_T V_j} \to \mathcal{F}_j|_{V_i \times_T V_j}$, which induce morphism of canonical descent datum for $V_i \times_T V_j \to U_i \cap U_j$, thus by lemma 3.2 it comes from a morphism $u_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$. Moreover in fact we have the following diagram:

$$\begin{tikzcd}
\mathcal{G}_i|_{V_i \times_T V_j} \arrow[r, "\psi_i|_{V_i \times_T V_j}"'] \arrow[d, "\varphi_{ij}"'] & \mathcal{F}_i|_{V_i \times_T V_j} \arrow[d, "u_{ij}|_{V_i \times_T V_j}"'] \\
\mathcal{G}_j|_{V_i \times_T V_j} \arrow[r, "\psi_j|_{V_i \times_T V_j}"'] & \mathcal{F}_j|_{V_i \times_T V_j}
\end{tikzcd}$$

Thus we see clearly the cocycle condition for ξ gives that of $(\coprod \mathcal{F}_i, u_{ij})$, and after pullback along $\coprod V_i \to \coprod U_i$, the $(\coprod \mathcal{F}_i, u_{ij})$ is pulled back to ξ. ∎

References

[1] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2018.