Flat Morphisms Revisited

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October 25, 2018

We mostly follow the treatment of flatness in the Stacks Project: [1, Tag 00HD] and [1, Tag 00MD]

Notation. For any local ring A we denote its maximal ideal by $\mathfrak{m}_A$.

Let A be a ring.

Definition 1. An A-module M is *flat (over A)* if the functor $M \otimes_A (\cdot)\colon (\mathrm{Mod}_A) \to (\mathrm{Mod}_A)$ is exact. An A-module M is *faithfully flat (over A)* if every complex of A-modules $N' \to N \to N''$ is exact if and only if $M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ is exact

A ring morphism $A \to B$ is *flat* if it makes B into a flat A-module. Similiarly, the ring morphism $A \to B$ is *faithfully flat* if it makes B into a faithfully flat A-module.

The following proposition was copied from Ole's handout for his talk on flatness in the seminar on moduli spaces.

Proposition 2. *(i) Every free A-module is flat.*

(ii) The tensor product of flat A-modules is a flat A-module.

(iii) If $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of A-modules with* M'' *flat, then the sequence stays exact after tensoring with any A-module.*

(iv) Let $0 \to M' \to M \to M'' \to 0$ *be short exact sequence of A-modules where* M'' *is flat. If one of the modules* M' *or* M *is flat, then all three are flat.*

(v) For any ring morphism $A \to B$ *and any flat A-module M the module* $M \otimes_A B$ *is flat over B.*

(vi) Suppose $A \to B$ *is a flat ring morphism. Then every flat B-module is a flat A-module.*

Proposition 3. *Suppose M is a flat A-module. Then the following are equivalent:*

(i) M is a faithfully flat A-module,

(ii) for all A-modules N if $M \otimes_A N = 0$*, then* $N = 0$*,*

(iii) for all prime ideals $\mathfrak{p} \subset B$ *the module* $M \otimes_B \kappa(\mathfrak{p})$ *is nonzero,*

(iv) for all maximal ideals $\mathfrak{m} \subset B$ *the module* $M \otimes_B \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ *is nonzero.*

Proof. The implications *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)* $\Longrightarrow$ *(iv)* are immediate.

To see the implication *(iv)* $\Longrightarrow$ *(i)* consider a complex $N' \to N \to N''$. Denote by H the homology of this complex. Since M is assumed to be flat, the homology $\tilde{H}$ of the tensored complex $M \otimes N' \to M \otimes N \to M \otimes N''$ is equal to $H \otimes M$. Assume the tensored complex is exact, i.e., $H \otimes M = 0$. Suppose by contradiction that $x \in H \setminus 0$. Consider the annihilator $\mathrm{Ann}(x) \subset A$. The inclusion $A/\mathrm{Ann}(x) \subset H$ yields the inclusion $M/\mathrm{Ann}(x)M \subset H \otimes M$ because M is flat. However, the ideal $\mathrm{Ann}(x)$ is contained in some maximal ideal $\mathfrak{m} \subset A$ and $0 = M/\mathrm{Ann}(x)M$ surjects onto $M/\mathfrak{m}M \neq 0$. Contradiction. □

Corollary 4. *(i) A flat ring morphism $A \to B$ is faithfully flat if and only if the associated morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

(ii) A flat ring morphism $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ *is in the image of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

(iii) Every flat morphism of local rings is faithfully flat.

Proof. The fiber over $\mathfrak{p} \in \mathrm{Spec}(A)$ is nonempty precisely when $B \otimes_A \kappa(\mathfrak{p}) \neq 0$. □

Definition 5. Let $f\colon X \to Y$ be a morphism of schemes. A quasicoherent sheaf $\mathcal{F}$ on X is *flat at the point $x \in X$ over Y* or *f-flat at the point $x \in X$* if $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,f(x)}$-module. We sat that $\mathcal{F}$ is *f-flat* if it is f-flat at x for every $x \in X$. The morphism $f\colon X \to Y$ is *flat (at x)* if $\mathcal{O}_X$ is f-flat (at x) (i.e., the local ring morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat).

A morphism of schemes is *faithfully flat* if it is flat and surjective.

The following proposition was also copied from Ole's handout and is a translation of Proposition 2 into the setting of schemes.

Proposition 6. *Let $f\colon X \to Y$ be a morphism of schemes.*

(i) Every locally free $\mathcal{O}_X$-module is flat over X.

(ii) The tensor product of f-flat $\mathcal{O}_X$-modules is f-flat.

(iii) If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is a short exact sequence of quasicoherent $\mathcal{O}_Y$-modules and $\mathcal{G}''$ is flat over Y, then the sequence stays exact after pulling back along f.

(iv) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasicoherent $\mathcal{O}_X$-modules where $\mathcal{F}''$ is f-flat. If one of $\mathcal{F}$ or $\mathcal{F}'$ is f-flat, then all three $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ are f-flat.

(v) For every cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

and f-flat quasicoherent sheaf $\mathcal{F}$ on X the sheaf $(g')^*\mathcal{F}$ is f'-flat.

(vi) Suppose $Y \to Z$ is a flat morphism. Then every quasicoherent sheaf $\mathcal{F}$ on X which is flat over Y is also flat over Z.

Theorem 7. *Let $f: X \to Y$ be a morphism locally of finite type and Y a locally noetherian scheme. If f is flat, then it is an open map.*

Lemma 8 (Going down for flat morphisms)**.** Suppose $A \to B$ is a flat ring morphism. Let $\mathfrak{p} \subset A$ be a prime ideal and $\mathfrak{q} \subset B$ a prime ideal lying over $\mathfrak{p}$. Then for every prime ideal $\mathfrak{p}' \subset \mathfrak{p}$ of A there exists a prime ideal $\mathfrak{q}' \subset \mathfrak{q}$ of B that lies over $\mathfrak{p}'$.

Proof of Theorem 7. Since openness is a local property, we reduce the affine case; we show that for a noetherian ring A and a finite type ring morphism $A \to B$, the map $f: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open.

Recall Chevalley's theorem: for every finite type morphism of noetherian schemes the image of any constructible set is constructible. In particular the image of f is constructible. Recall also that a constructible set in a noetherian topological space is open if and only if it is stable under generization. Lemma 8 translates to: the image of f is stable under generization. □

Flatness Criteria

Let A be a ring and M an A-module. In general the functor $M \otimes_A (\cdot)$ is right exact. We want to measure its failure to be exact, that is, we want to define a derived functor. Since every A-module admits a free resolution, the category (Mod_A) has enough projectives.

Definition 9. Define $\mathrm{Tor}_\bullet^A : (\mathrm{Mod}_A) \to (\mathrm{Mod}_A)$ to be the left derived δ-functor of $M \otimes_A (\cdot)$.

Proposition 10 (Flatness through Tor)**.** *The following are equivalent:*

(i) *M is a flat A-module,*

(ii) $\mathrm{Tor}_i^A(M, N) = 0$ *for all A-modules N and $i > 0$,*

(iii) $\mathrm{Tor}_1^A(M, N) = 0$ *for all A-modules N.*

Theorem 11 (Ideal-theoretic criterion)**.** *An A-module M is flat if and only if* $\mathrm{Tor}_1^A(M, A/\mathfrak{a}) = 0$ *for all ideals $\mathfrak{a} \subset A$.*

Sketch of Proof. Suppose that $\mathrm{Tor}_1^A(M, A/\mathfrak{a}) = 0$ for all ideals $\mathfrak{a} \subset A$. Let N be a finitely generated A-module. We will show that $\mathrm{Tor}_1^A(M, N) = 0$. Assume first that N is a finitely generated, say by the elements $x_1, \ldots, x_n$, over A. We induct on the number of generators n. The key idea is to consider the annihilator $\mathrm{Ann}(x_n) \subset A$. Next we look at the long exact seqence in $\mathrm{Tor}_\bullet^A(M, \cdot)$ evaluated at the short exact sequence $0 \to A/\mathrm{Ann}(x_n) \to N \to Q \to 0$. The module Q is generated by the elements $x_1, \ldots, x_{n-1}$, so by induction hypothesis $\mathrm{Tor}_1^A(M, Q) = 0$. By assumption $\mathrm{Tor}_1^A(M, A/\mathrm{Ann}(x_n)) = 0$. We conclude that $\mathrm{Tor}_1^A(M, N) = 0$, because it is stuck between two zeros in an exact sequence. For general N: write N as the colimit of finitely generated A-submodules and use that homology commutes with colimits. □

The long exact sequence in $\operatorname{Tor}_\bullet^A(M, \cdot)$ associated to the short exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ starts off as

$$0 \longrightarrow \operatorname{Tor}_1^A(M, A/\mathfrak{a}) \longrightarrow \mathfrak{a} \otimes_A M \longrightarrow M \longrightarrow M/\mathfrak{a}M \longrightarrow 0.$$

Thus we can restate the ideal-theoretic flatness criterion as: M is a flat A-module if and only if the map $\mathfrak{a} \otimes_A M \to M$ is injective for all ideals $\mathfrak{a} \subset A$.

Corollary 12 (Equational criterion)**.** *An A module M is flat if and only if every relation in M is trivial, i.e., for every relation $\sum_i f_i x_i = 0$ in M there are elements $y_j \in M$ and elements $a_{ij} \in A$ such that $x_i = \sum_j a_{ij} y_j$ for all i and $\sum_i f_i a_{ij} = 0$ for all j.*

Intuitively, a relation in M is trivial if it is secretly a relation in A.

Proof. First assume that every relation in M is trivial. Let $\mathfrak{a} \subset A$ be an ideal. Let $x = \sum_i f_i \otimes x_i$ be an element in $\ker(\mathfrak{a} \otimes M \to M)$, that is, $\sum_i f_i x_i = 0$ is a relation in M and so must be trivial. We compute

$$x = \sum_i f_i \otimes x_i = \sum_i f_i \otimes (\sum_j a_{ij} y_j) = \sum_j (\sum_i f_i a_{ij}) \otimes y_j = \sum_i 0 \otimes y_j = 0.$$

Hence $\operatorname{Tor}_1^A(M, A/\mathfrak{a}) = \ker(\mathfrak{a} \otimes M \to M) = 0$. We conclude by Theorem 11 that M is flat.

Now assume that M is flat. Let $\sum_{i=1}^n f_i x_i = 0$ be a relation in M. Consider the ideal $\mathfrak{a} \subset A$ generated by the f_i . We have a short exact sequence

$$0 \longrightarrow K \longrightarrow A^{\oplus n} \xrightarrow{e_i \mapsto f_i} \mathfrak{a} \longrightarrow 0.$$

We also have the inclusion $\mathfrak{a} \hookrightarrow A$. Tensoring these two diagrams with M and splicing them together we get the following diagram with exact column(s) and row(s).

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & K \otimes M & \longrightarrow & (A \otimes M)^{\oplus n} & \longrightarrow & \mathfrak{a} \otimes M & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & M. & & \end{array}$$

The element $\sum_i f_i x_i = 0 \in M$ is the image of $\sum_i f_i \otimes x_i \in \mathfrak{a} \otimes M$. By injectivity we have $\sum_i f_i \otimes x_i = 0$. Therefore $\sum_i e_i \otimes x_i$ maps to 0, so we can write it as an element $\sum_j k_j \otimes y_j \in K \otimes M$. Since the e_i form a basis of $A^{\oplus n}$ we can write $k_j = \sum_i a_{ij} e_i$ for some $a_{ij} \in A$. We conclude that $\sum_i f_i x_i = 0$ is a trivial relation. □

For finitely generated modules over a local ring we only need to check $\operatorname{Tor}_\bullet$-acyclicity for the residue field.

Theorem 13 (Local criterion)**.** *Suppose $A \to B$ is a morphism of noetherian local rings and M is a finitely generated B-module. Then M is A-flat if and only if $\operatorname{Tor}_1^A(M, A/\mathfrak{m}_A) = 0$.*

Sketch of Proof. Set $\mathfrak{m} := \mathfrak{m}_A$. Suppose $\operatorname{Tor}_1^A(M, A/\mathfrak{m}) = 0$.

Lemma 14. For all A-modules N of finite length we have $\operatorname{Tor}_1^A(M, N) = 0$.

Consider the inclusion of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}\cap\mathfrak{m}^n & \longrightarrow & \mathfrak{a}\oplus\mathfrak{m}^n & \longrightarrow & \mathfrak{a}+\mathfrak{m}^n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A\oplus A & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Tensor the M to obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccc} \operatorname{Tor}_1^A(M, A/(\mathfrak{a}\cap\mathfrak{m}^n)) & \longrightarrow & \operatorname{Tor}_1^A(M, A/\mathfrak{a})\oplus 0 & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ (\mathfrak{a}\cap\mathfrak{m}^n)\otimes_A M & \xrightarrow{\phi_n} & \mathfrak{a}\otimes_A M\oplus\mathfrak{m}^n\otimes_A M & \longrightarrow & (\mathfrak{a}+\mathfrak{m}^n)\otimes_A M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & \longrightarrow & M\oplus M & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

The zeros in the first row come from the finite colength of the ideals $\mathfrak{m}^n$ and $\mathfrak{a}+\mathfrak{m}^n$ ($0 = \mathfrak{m}^n/\mathfrak{m}^n \subset \mathfrak{m}^{n-1}/\mathfrak{m}^n \subset \ldots \subset \mathfrak{m}/\mathfrak{m}^n \subset A/\mathfrak{m}^n$ is a composition series for $A/\mathfrak{m}^n$).

Set $K := \operatorname{Tor}_1^A(M, A/\mathfrak{a})$. The diagram shows that K is contained in the iamge of ϕ_n.

By Artin-Rees, we have the inclusion $\mathfrak{a}\cap\mathfrak{m}^n \subset \mathfrak{m}^r\mathfrak{a}$ for all $r > 0$ and for all $n \gg 0$. The submodule $\mathfrak{m}^r(\mathfrak{a}\otimes_A M) \subset \mathfrak{a}\otimes_A M$ is the image of $\mathfrak{m}^r\mathfrak{a}\otimes_A M$. In particular, the image $\operatorname{im}(\phi_n)$ is contained in $\mathfrak{m}^r(\mathfrak{a}\otimes_A M)$ for all $n \gg 0$. Altogether, we obtain the inclusion

$$K \subset \bigcap_{r>0} \mathfrak{m}^r(\mathfrak{a}\otimes_A M) = 0;$$

the intersection is zero by Krull's Intersection Theorem.

Therefore $\operatorname{Tor}_1^A(M, A/\mathfrak{a}) = 0$ and M is flat by Theorem 11. □

Corollary 15 (Variant of the local criterion)**.** *Let $A \longrightarrow B$ be a local ring morphism of noetherian local rings. Let $\mathfrak{a} \subset A$ be an ideal in A and let M be a finitely generated B-module. Suppose that $M/\mathfrak{a}M$ is flat over $A/\mathfrak{a}$. Then M is flat over A if and only if $\operatorname{Tor}_1^A(M, A/\mathfrak{a}) = 0$.*

Proof. By the local criterion, Theroem 13, it suffices to show that $\mathfrak{m}_A \otimes_A M \longrightarrow M$ is injective.

Let $\sum_i f_i \otimes x_i \in \ker(\mathfrak{m}_A \otimes M \longrightarrow M)$. Applying the equational criterion (Corollary 12) to the relation $\sum_i f_i x_i = 0$ in the flat $A/\mathfrak{a}$-module $M/\mathfrak{a}M$, we find elements $a_{ij} \in A$ and $y_j \in M$ such that

$$x_i = \sum_j a_{ij} y_j \mod \mathfrak{a}M,$$
$$0 = \sum_i f_i a_{ij} \mod \mathfrak{a}.$$

We calculate

$$\sum_i f_i \otimes x_i = \sum_i f_i \otimes x_i + \sum_{i,j} f_i a_{ij} \otimes y_j - \sum_{i,j} f_i a_{ij} \otimes y_j$$
$$= \sum_i f_i \otimes (x_i - \sum_j a_{ij} y_j) + \sum_j (\sum_i f_i a_{ij}) \otimes y_j$$

Since $x_i - \sum_j a_{ij} y_j \in \mathfrak{a}M$ and $\sum_i f_i a_{ij} \in \mathfrak{a}$, it follows that $\sum_i f_i \otimes x_i$ is in the image of the map $\mathfrak{a} \otimes_A M \to \mathfrak{m}_A \otimes_A M$.

In particular, all elements in $\ker(\mathfrak{m}_A \otimes_A M \to M)$ are images of elements in $\ker(\mathfrak{a} \otimes_A M \to M)$. Note that the map $\mathfrak{a} \otimes_A M \to M$ is injective, because we assume $\mathrm{Tor}_1^A(M, A/\mathfrak{a}) = 0$. Hence $\ker(\mathfrak{m}_A \otimes_A M \to M) = 0$. □

Theorem 16 (Fiberwise criterion, local ring version)**.** *Let $A \to B \to C$ be morphisms of noetherian local rings. Suppose M is a nonzero finitely generated C-module which is flat over A and such that $M/\mathfrak{m}_A M$ is a flat $B/\mathfrak{m}_A B$-module. Then $A \to B$ is a flat ring morphism and M is flat over B.*

Sketch of Proof. Step 1. We show that M is faithfully flat. Let $\mathfrak{b} := \mathrm{im}(\mathfrak{m}_A \otimes_A B \to B) = \mathfrak{m}_A B$. The map $\mathfrak{m}_A \otimes_A M \twoheadrightarrow \mathfrak{b} \otimes_B M$ is surjective and the composition $\mathfrak{m}_A \otimes_A M \to \mathfrak{b} \otimes_B M \to M$ is injective. Hence $\mathfrak{b} \otimes_B M \to M$ is injective (i.e., $\mathrm{Tor}_1^B(M, B/\mathfrak{b}) = 0$), so M is flat by Corollary 15. Then it follows by Nakayama that M is faithfully flat.

Step 2. We tensor the short exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(B, \kappa(A)) \longrightarrow \mathfrak{m}_A \otimes_A B \longrightarrow \mathfrak{b} \longrightarrow 0.$$

with M. Use the injectivity of $\mathfrak{m}_A \otimes_A M \to \mathfrak{b} \otimes_B M$ and the faithful flatness of M over B to conculde that $\mathrm{Tor}_1^A(B, \kappa(A)) = 0$. □

Theorem 17 (Fiberwise criterion, scheme version)**.** *Let S be a locally noetherian scheme. Let $f : X \to Y$ be a morphism of locally noetherian S-schemes and $\mathcal{F}$ a nonzero coherent $\mathcal{O}_X$-module on X. Let $x \in X$. Let $y := f(x)$ and let $s \in S$ be the image of x in S. Then the following are equivalent:*

(i) $\mathcal{F}$ is flat over S at x and $\mathcal{F}_s$ is flat over Y_s at x,

(ii) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Remark 18. Most of the noetherian hypotheses can be replaced with locally of finite presentation hypotheses.

References

[1] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2018.