Étale morphisms

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We mostly follow Bhatt’s notes [1].

**Definition.** A local homomorphism of local rings \( f : (B, n) \to (A, m) \) is called **unramified** if \( f(n)B = m \) and \( \kappa(m) \) is a finite separable extension of \( \kappa(n) \).

**Definition.** A morphism of schemes \( \pi : X \to Y \) is called **unramified** at \( x \in X \) if

(i) the local homomorphism \( O_{Y, f(x)} \to O_{X, x} \) is unramified,

(ii) \( \pi \) is locally of finite type at \( x \).

If \( \pi \) is unramified at all \( x \in X \), it is called **unramified**.

**Lemma 1.** Suppose \( A \) is a finitely generated algebra over a field \( k \) with \( \Omega_{A/k} = 0 \). Then \( A \) is a finite direct sum of finite separable field extensions of \( k \).

**Sketch of proof.** First assume \( k = \bar{k} \). Then for any prime \( p \subset A \) and any maximal ideal \( m \subset A \) containing \( p \),

\[
m_m / m_m^2 \cong k \otimes_{A_m} \Omega_{A_m/k} = 0.
\]

By Nakayama’s lemma, it follows that \( m_m = p_m = 0 \). Varying \( p \) and \( m \), we deduce that \( A \) is a reduced artinian \( k \)-algebra, hence a finite direct sum of copies of \( k \).

Deduce the case of arbitrary \( k \) using a base change

\[
A \otimes_k \bar{k} \xleftarrow{\square} A \quad \xrightarrow{\square} \quad \bar{k} \xleftarrow{\square} k.
\]

**Theorem 2.** Suppose \( \pi : X \to Y \) is locally of finite type. Then for any \( x \in X \), the following are equivalent:

(i) \( \pi \) is unramified at \( x \).

(ii) \( \Omega_{X/Y, x} = 0 \).

(iii) There exists an open \( x \in U \) and a locally closed embedding \( j : U \to \mathbb{A}_Y^n \) defined by an ideal sheaf \( I \) such that \( \Omega_{\mathbb{A}_Y^n / Y, x} \) is generated by \( dg \) for sections \( g \) of \( I \).

(iv) There exists an open \( x \in U \) such that \( \text{diag}_{X/Y} |_U \) is an open embedding.
**Sketch of proof.** (i) $\implies$ (ii). Consider a homomorphism $B \to A$ and primes $p \in \text{Spec } A$, $q = p \cap B$. If $B_q \to A_p$ is an unramified homomorphism of local rings, we have a cartesian diagram

\[
\begin{array}{c}
\kappa(p) \\ \uparrow \\
\kappa(q) \\ \uparrow \\
A_p \\ \square \\ B_q
\end{array}
\]

It follows that

\[
\Omega_{A/B_p} \otimes_{A_p} \kappa(p) = \Omega_{A_p/B_q} \otimes_{A_p} \kappa(p) = \Omega_{\kappa(p)/\kappa(q)} = 0.
\]

(ii) $\implies$ (i). Use Lemma 1.

(ii) $\iff$ (iii). Use the conormal exact sequence

\[
j^*(I/I^2) \to j^*\Omega_{A^n/Y} \to \Omega_{U/Y} \to 0.
\]

(ii) $\iff$ (iv). We show that for any affine opens Spec $B \subset Y$ and Spec $A \subset \pi^{-1}(\text{Spec } B)$, the closed embedding Spec $A \to \text{Spec } A \otimes_B A$ is actually an open embedding if and only if $\Omega_{A/B} = 0$. To this end, apply the following lemma to the ideal ker $(A \otimes_B A \to A)$.

**Lemma 3.** Suppose $R$ is a ring and $I \subset R$ is a finitely generated ideal. If $I^2 = I$, then $V(I) = D(e)$ for an idempotent element $e \in R$.

**Proposition 4.** Unramified morphisms are stable under base change and composition. A morphism that is locally of finite type is unramified if and only if all its fibers are unramified.

**Sketch of proof.** Use Theorem 2.(ii).

**Proposition 5.** (i) If for morphisms $f : X \to Y$ and $g : Y \to Z$ the composition $gf$ is unramified, then so is $f$.

(ii) Every monomorphism locally of finite type is unramified.

**Sketch of proof.** (i) Use Theorem 2.(ii).

(ii) Use Theorem 2.(iv).

**Theorem 6.** Suppose $\pi : X \to S$ is locally of finite type. Then $\pi$ is unramified if and only if for every affine morphism $Y \to S$ and every closed subscheme $Y_0 \subset Y$ defined by an ideal sheaf $I$ with $I^2 = 0$, the map

\[
\text{Mor}_S(Y, X) \to \text{Mor}_S(Y_0, X)
\]

is injective.
Sketch of proof. Reduce to the affine case as in the following diagrams:

\[\begin{align*}
A 
\downarrow 
\to 
R 
\to 
B 
\to 
B/I 
\end{align*}\]

\[\begin{align*}
X 
\uparrow 
\to 
S 
\to 
Y 
\to 
Y_0 
\end{align*}\]

Fix a homomorphism \(A \to B/I\). The trick is to notice that differences of factorizations \(A \to B\) correspond to derivations \(A \to I\).

For the backward implication, consider \(B := (A \otimes_R A)/J^2\), where \(J = \ker(A \otimes_R A \to A)\), as well as the ideal \(I := J/J^2\).

Definition. A morphism of schemes \(\pi : X \to Y\) is called \(\acute{e}tale\) at \(x \in X\) if \(\pi\) is unramified and flat at \(x\).

It is called \(\acute{e}tale\) if it is \(\acute{e}tale\) at all points \(x \in X\).

Proposition 7. Consider morphisms \(f : X \to Y\) and \(g : Y \to Z\). If \(g\) is unramified and \(gf\) is \(\acute{e}tale\), then \(f\) is \(\acute{e}tale\).

Sketch of proof. Use Proposition 5 and the fiberwise criterion for flatness, Theorem 17 of Sebastian’s talk.

Theorem 8. A morphism \(\pi : X \to Y\) is \(\acute{e}tale\) if and only if the following holds:

(i) There exists an open \(x \in U\) and a locally closed embedding \(j : U \hookrightarrow \mathbb{A}_Y^n\).

(ii) If \(I\) is the corresponding ideal sheaf, then there exist sections \(g_1, \ldots, g_n\) of \(I\) such that the \(d g_1, \ldots, d g_n\) form a basis for \(\Omega_{\mathbb{A}_Y^n/Y, x} \otimes \mathcal{O}_{\mathbb{A}_Y^n, x} \kappa(x)\).

Sketch of proof. (ii) \(\implies\) (i). Unramifiedness follows from Theorem 2. Flatness uses the theory of Cohen-Macaulay rings. See for example the exposition in [3, Section 25.2.1].

(i) \(\implies\) (ii). See for example [2, Tag 00UE].

We record some more properties of \(\acute{e}tale\) morphisms that follow quickly from the properties of unramified and flat morphisms:

Proposition 9. \(\acute{e}tale\) morphisms are open.

Sketch of proof. In fact, flat morphisms locally of finite type are open (Theorem 7 of Sebastian’s talk).

Proposition 10. \(\acute{e}tale\) morphisms are stable under base change and composition.

Proposition 11. \(\acute{e}tale\) morphisms are quasi-finite.

Proposition 12. A flat morphism locally of finite type is \(\acute{e}tale\) if and only if it is unramified.

Finally, we present an analog of Theorem 6 for \(\acute{e}tale\) morphisms:
Theorem 13. Suppose $\pi : X \to S$ is locally of finite type and separated. Then $\pi$ is étale if and only if for every affine morphism $Y \to S$ and every closed subscheme $Y_0 \subset Y$ defined by an ideal sheaf $I$ with $I^2 = 0$, the map

$$\text{Mor}_S(Y, X) \to \text{Mor}_S(Y_0, X)$$

is bijective.

Sketch of proof. (i) $\implies$ (ii). Reduce to the case $Y = S$ via base change. Let $\varphi : X \to Y$ be the structure morphism. Then the cartesian diagram

$$\begin{array}{ccc}
Y & \xrightarrow{s} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{diag}_{X/Y}} & X \times_Y X
\end{array}$$

shows that every section $s : Y \to X$ is an isomorphism onto a connected component of $X$. Now consider a morphism $t \in \text{Mor}_Y(Y_0, X)$. Since the underlying sets of $Y_0$ and $Y$ are the same, there is a connected component $X_i$ of $Y$ such that $X_i \to Y$ is a universal homeomorphism. Also, $X_i \to Y$ is étale. Now the faithfully flat base change

$$\begin{array}{ccc}
X_i \times_Y X_i & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & Y
\end{array}$$

shows that $X_i \to Y$ is in fact an isomorphism. The inverse is our desired extension of $t$ to $Y$.

(ii) $\implies$ (i). Assume $S = \text{Spec } R$, $X = \text{Spec } R[X]/I$. Using the hypothesis, find a splitting of the exact sequence

$$0 \to I/I^2 \to R[X]/I^2 \to R[X]/I \to 0.$$ 

The resulting map $R[X]/I^2 \to I/I^2$ is a derivation, so induces a map $\Omega_{R[X]/R} \otimes_{R[X]} R[X]/I \to I/I^2$ that is inverse to $I/I^2 \to \Omega_{R[X]/R} \otimes_{R[X]} R[X]/I$. $\blacksquare$

References

