Flat and étale morphisms

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All rings are commutative.

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1 Flat morphisms

1.1 Preliminaries on tensor product

Let $A$ be a ring, $M$ and $A$-module. For all $A$-modules $N_1, N_2$ we have a natural isomorphism

$$\text{Hom}_A(N_1 \otimes_A M, N_2) \cong \text{Hom}_A(N_1, \text{Hom}_A(M, N_2)).$$

In other words $\otimes_A M$ is left adjoint to $\text{Hom}_A(M, -)$. Hence $\otimes_A M$ is right exact and commutes with colimits.
Left derived functors $L^i(\otimes_A M)(-)$ are denoted $\text{Tor}^A_i(-, M)$. A morphism of modules $M \to M'$ induces natural morphisms $\text{Tor}^A_i(-, M) \to \text{Tor}^A_i(-, M')$, so $\text{Tor}_i$ is a bifunctor. The most important property of Tor is its commutativity:

**Theorem 1.1.1.** Let $A$ be a ring, and let $M, N$ be $A$-modules. For every $i \geq 0$ there exists a natural isomorphism $\text{Tor}^A_i(N, M) \to \text{Tor}^A_i(M, N)$\footnote{See [3], chapter 2, section 2.7}.

We will not need the full force of this theorem and so omit its proof.

**Proposition 1.1.2.** Let $A$ be a ring, $I \subset A$ an ideal, and $M$ an $A$-module. $\text{Tor}^A_1(A/I, M) = \ker (I \otimes_A M \to M)$.

**Proof.** The short exact sequence $0 \to I \to A \to A/I \to 0$ induces an exact sequence $0 = \text{Tor}^A_1(A, M) \to \text{Tor}^A_1(A/I, M) \to I \otimes_A M \to M$. □

**Corollary 1.1.3.** Let $a \in A$ be a nonzero element. $\text{Tor}^A_1(A/(a), M)$ is the $a$-torsion of $M$.

Let $A, B$ be rings, $N_1$ an $A$-module, $N_2$ an $A, B$-bimodule, and $N_3$ a $B$-module. There is an isomorphism of $A, B$-bimodules

$$(N_1 \otimes_A N_2) \otimes_B N_3 \to N_1 \otimes_A (N_2 \otimes_B N_3),$$

which is natural in $N_1, N_2, N_3$.

Also recall that if $A$ is a ring and $S \subset A$ a multiplicative system, then the functor $\otimes_A A_S$ is isomorphic to the functor of localization at $S$.

### 1.2 Flat modules

**Definition 1.2.1.** Let $A$ be a ring. A module $M$ over $A$ is called flat if $\otimes_A M$ is exact.

**Proposition 1.2.2.** Let $A \to B$ be a morphism of rings, and $M$ a $B$-module. If $M$ is flat over $B$ and $B$ is flat over $A$ then $M$ is flat over $A$.

**Proof.** The functor $- \otimes_A M$ is isomorphic to the composition $(- \otimes_A B) \otimes_B M$ of exact functors. □

**Proposition 1.2.3.** Let $A \to B$ be a morphism of rings. If $M$ is a flat $A$-module, then $B \otimes_A M$ is a flat $B$-module.

**Proof.** The functor $- \otimes_B (B \otimes_A M)$ is isomorphic to the functor $- \otimes_A M$, which is exact. □
Localization is exact. Hence, if \( \otimes M \) of \( A \) proof.

The “only if” part is trivial. We want to show that for arbitrary inclusion
of \( B \)-module inherited from \( M \). Let \( q \in \text{Spec} B \), and \( p = \varphi^{-1}q \). We have
an isomorphism of functors from the category of \( A \)-modules to the category of
\( B_q \)-modules:

\[
(- \otimes A M)_q = (- \otimes A M) \otimes_B B_q = - \otimes_A (M \otimes_B B_q) = - \otimes_A M_q = - \otimes_A (A_p \otimes_{A_p} M_q) = (-) p \otimes_{A_p} M_q.
\]

Notice that \( \otimes A M \) of \( A \) over \( N \) exact sequence

\[
\text{Proposition 1.2.5.} \text{ Let } A \text{ be a ring. An } A \text{-module is flat if and only if } I \otimes_A M \to M \text{ is injective (equivalently, } \text{Tor}_1^A(A/I, M) = 0) \text{ for every finitely generated ideal } I \subset A.
\]

\[
\text{Proof.} \text{ The “only if” part is trivial. We want to show that for arbitrary inclusion of } A \text{-modules } N' \subset N \text{ the induced morphism } N' \otimes_A M \to N \otimes_A M \text{ is injective.}
\]

We first show that \( I \otimes_A M \to M \) is injective for every ideal \( I \). Let \( x \in I \otimes_A M \) be an element which vanishes in \( M \). The element \( x \) is a finite linear combination of elementary tensors \( y \otimes m \) where \( y \in I, m \in M \). Thus there exists a finitely generated ideal \( I' \subset I \) and \( x' \in I' \otimes_A M \) such that the image of \( x' \) in \( I' \otimes_A M \) is equal to \( x \). The map \( I' \otimes_A M \to M \) is injective, so \( x' = 0 \) and hence \( x = 0 \), i.e. \( I \otimes_A M \to M \) is injective. As a corollary, \( \text{Tor}_1^A(N, M) = 0 \) if \( N \) is a cyclic module, that is, \( N = A/I \) for some ideal \( I \subset A \).

Let \( N \) be an arbitrary module and \( N' \) its submodule. Consider an index set \( J \) whose elements are finite subsets of \( N \setminus N' \). For \( j \in J \) let \( N_j \) be the submodule of \( N \) generated by \( N' \) and \( j \). If \( j \subset j' \) then there is a natural injection \( N_j \to N_{j'} \).

The inclusion order on \( J \) makes it a directed poset. Clearly, \( \text{colim}_{j \in J} N_j = N \).

Let \( j \subset j' \) be an inclusion. Assume that \( j' \setminus j \) consists of a single element. In this case \( N_{j'}/N_j \) is a cyclic module. The short exact sequence \( 0 \to N_j \to N_{j'}/N_j \to 0 \) induces an exact sequence \( \text{Tor}_1^A(N_{j'}/N_j, M) \to N_{j'} \otimes_A M \to N_j \otimes_A M \). Since \( N_{j'}/N_j \) is cyclic, \( \text{Tor}_1^A(N_{j'}/N_j, M) \) vanishes, and so \( N_j \otimes_A M \to N_{j'} \otimes_A M \) is injective.

A general inclusion \( j \subset j' \) can be factored into a sequence of inclusions such that at each step only one new element appears. Hence \( N_j \otimes_A M \to N_{j'} \otimes_A M \) is injective, which implies that the morphism \( N' \otimes_A M \to \text{colim}_{j \in J} N_j \otimes_A M \) is injective too. It remains to recall that \( \otimes_A M \) commutes with colimits. \( \square \)
Corollary 1.2.6. Let $A$ be a PID. An $A$-module $M$ is flat if and only if it is torsion-free.

Proposition 1.2.7. Let $A$ be a ring, let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence of $A$-modules, and let $N$ be an $A$-module. If $M$ is flat then $M' \otimes_A N \to M'' \otimes_A N$ is injective.

Proof. One can either refer to commutativity of Tor or do a direct proof as follows. Let $0 \to K \to F \to N \to 0$ be a short exact sequences with $F$ a free module. Consider a commutative diagram with exact rows and columns:

$$
\begin{array}{cccccc}
0 & & & & & \\
\downarrow & & & & & \\
M' \otimes_A K & \longrightarrow & M'' \otimes_A K & \longrightarrow & M \otimes_A K & \longrightarrow & 0 \\
\downarrow & & & & & \\
0 & \longrightarrow & M' \otimes_A F & \longrightarrow & M'' \otimes_A F & \longrightarrow & M \otimes_A F & \longrightarrow & 0 \\
\downarrow & & & & & \\
M' \otimes_A N & \longrightarrow & M'' \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0. \\
\downarrow & & & & & \\
0 & & & & & 0.
\end{array}
$$

A simple diagram chase finishes the proof.

Theorem 1.2.8. Let $A$ be a local noetherian ring, and $M$ an $A$-module of finite type. If $M$ is flat then it is free.

Proof. Let $k$ be the residue field of $A$. Take a $k$-basis of $M \otimes_A k$. Lifting it to $M$ we obtain a morphism from a free $A$-module $F$ of finite type to $M$. By Nakayama lemma this morphism is surjective. Let $K$ be its kernel. Tensoring the short exact sequence $0 \to K \to F \to M \to 0$ by $k$ we obtain exact sequence $K \otimes_A k \to F \otimes_A k \to M \otimes_A k \to 0$. The morphism $K \otimes_A k \to F \otimes_A k$ is injective by proposition [1.2.7]. The morphism $F \otimes_A k \to M \otimes_A k$ is an isomorphism by construction. Hence $K \otimes_A k$ is zero. On the other hand, $K$ is of finite type since $A$ is noetherian. So, Nakayama lemma shows that $K = 0$.

1.3 Artin-Rees lemma and Krull intersection theorem

Let $A$ be a ring, $I \subset A$ an ideal.

Definition 1.3.1. Let $M$ be an $A$-module. An $I$-filtration on $M$ is a descending chain of submodules $F_iM \subset M$, $i \in \mathbb{Z}_{\geq 0}$, such that $F_0M = M$ and $IF_iM \subset F_{i+1}M$ for every $i$. 

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Definition 1.3.2. Let \( M \) be an \( A \)-module. An \( I \)-filtration \( F_iM \) is called stable if \( IF_iM = F_{i+1}M \) for sufficiently large \( i \).

Proposition 1.3.3. Let \( A \) be a ring, \( I \subset A \) an ideal, and let \( N, M \) be \( A \)-modules. If \( F_iN \) is a stable \( I \)-filtration of \( N \) then the filtration of \( N \otimes_A M \) by images of \( F_iN \otimes_A M \) is stable.

Proof. Omitted.

Proposition 1.3.4. Let \( A \to B \) be a morphism of rings, \( I \subset A \) an ideal, and \( M \) a \( B \)-module, and \( F_iM \) a stable \( I \)-filtration of \( M \) as an \( A \)-module. If each \( F_iM \) is a \( B \)-submodule, then \( F_iM \) is a stable \( IB \)-filtration of \( M \) as a \( B \)-module.

Proof. Omitted.

Let \( M \) be an \( A \)-module endowed with an \( I \)-filtration \( F_iM \). Consider a graded ring \( B_I A = \bigoplus_{i=0}^{\infty} I^i \) and a \( B_I A \)-module \( B_F M = \bigoplus_{i=0}^{\infty} F_iM \).

Proposition 1.3.5. Let \( A \) be a noetherian ring, \( I \subset A \) an ideal, \( M \) an \( A \)-module with an \( I \)-filtration \( F_iM \). The filtration is stable if and only if \( B_F M \) is of finite type over \( B_I A \).

Lemma 1.3.6 (Artin-Rees lemma). Let \( A \) be a noetherian ring, \( I \subset A \) an ideal, \( M \) an \( A \)-module with a stable \( I \)-filtration \( F_iM \), and \( N \subset M \) a submodule. The filtration \( F_iN = N \cap F_iM \) is stable.

Proof. The ring \( B_I A \) is noetherian since it is a quotient of the polynomial ring \( A[x_1, \ldots, x_n] \) for some \( n \). The module \( B_F N \) is a submodule of \( B_F M \), and thus is of finite type. Now the claim follows from the previous proposition.

Theorem 1.3.7 (Krull intersection theorem). Let \( A \) be a noetherian local ring, \( I \subset A \) an ideal and \( M \) a module of finite type. If \( F_iM \) is a stable \( I \)-filtration of \( M \), then \( \bigcap_{i=0}^{\infty} F_iM = 0 \).

Proof. Consider the submodule \( N = \bigcap_{i=0}^{\infty} F_iM \). By construction \( N \cap F_iM = N \) for every \( i \), and so by Artin-Rees lemma \( N = IN \). Hence \( N = mN \). Since \( N \) is of finite type, Nakayama lemma implies that \( N = 0 \).

1.4 Modules of finite length

Let \( A \) be a ring, \( M \) a module. A strict chain of submodules of length \( n \) is an increasing sequence of submodules of \( M \):

\[
M_0 \subset M_1 \subset \cdots \subset M_n,
\]
such that \( M_0 = 0, M_n = M \), and each inclusion \( M_i \subset M_{i+1} \) is nontrivial.

We define \( l_A(M) \), the length of \( M \), as the supremum of lengths of strict chains.
Definition 1.4.1. M is called a module of finite length if $l_A(M)$ is finite (i.e. if the supremum exists).

Proposition 1.4.2. $l_A(M) = 1$ if and only if $M = A/m$ for some $m \in \text{Spec}_{\text{max}} A$.

Proof. Excercise. \hfill \Box

Proposition 1.4.3. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of $A$-modules. If $M$ is of finite length or $M'$ and $M''$ are of finite length then all three modules are of finite length and $l_A(M) = l_A(M') + l_A(M'')$.

Proof. Excercise. \hfill \Box

Proposition 1.4.4. Let $A$ be a ring, $m \subset A$ a maximal ideal of finite type, and $M$ an $A$-module of finite type. If $m^n M = 0$ for some $n > 0$, then $M$ is of finite length.

Proof. Let $n > 0$ be an integer. Suppose that $A/m^n$ is of finite length. If a module $M$ of finite type is annihilated by $m^n$ then it is an $A/m^n$-module, and so is a quotient of a finite direct sum of $A/m^n$'s. Hence $M$ is of finite length.

We next prove that $A/m^n$ is of finite length using induction over $n$. The case $n = 1$ was already established. Consider a short exact sequence

$$0 \to m/m^n \to A/m^n \to A/m \to 0.$$ 

The module $m/m^n$ is of finite type since $m$ is, and is annihilated by $m^{n-1}$, whence of finite length. But then $A/m^n$ is also of finite length. \hfill \Box

Proposition 1.4.5. Let $A$ be a ring, $M$ an $A$-module. If $\text{Tor}^1_A(A/m, M) = 0$ for every $m \in \text{Spec}_{\text{max}} A$, then $\text{Tor}^1_A(N, M) = 0$ for every module $N$ of finite length.

Proof. We will do it by induction on $l_A(N)$. If $l_A(N) = 1$ then $N$ is of the form $A/m$, and so $\text{Tor}^1_A(N, M) = 0$ by assumption. Otherwise there exists a proper nontrivial submodule $N' \subset N$. Consider an exact sequence $\text{Tor}^1_A(N', M) \to \text{Tor}^1_A(N, M) \to \text{Tor}^1_A(N/N', M)$ induced by short exact sequence $0 \to N' \to N \to N/N' \to 0$. Since $l_A(N') < l_A(N)$ and $l_A(N/N') < l_A(N)$, we see that $\text{Tor}^1_A(N', M) = \text{Tor}^1_A(N/N', M) = 0$, so $\text{Tor}^1_A(N, M) = 0$. \hfill \Box

1.5 Criteria of flatness

Theorem 1.5.1 (Critère local de platitude). Let $A \to B$ be a local morphism of noetherian local rings, $k$ the residue field of $A$, and $M$ a $B$-module of finite type. If $\text{Tor}^1_A(k, M) = 0$ then $M$ is flat over $A$. 

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Proof. We want to show that for every ideal \( I \subset A \) the module \( \text{Tor}^A_1(A/I, M) \) vanishes. Notice that if \( A/I \) is of finite length, then \( \text{Tor}^A_1(A/I, M) = 0 \) by proposition 1.4.5.

Let \( m \subset A \) be the maximal ideal, and \( I \subset A \) an arbitrary ideal. Let \( n > 0 \) be an integer. Consider a diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I \cap m^n & \rightarrow & I & \rightarrow & I/(I \cap m^n) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & m^n & \rightarrow & A & \rightarrow & A/m^n & \rightarrow & 0.
\end{array}
\]

Tensoring it with \( M \) over \( A \) we obtain a diagram

\[
\begin{array}{ccccccccc}
(I \cap m^n) \otimes_A M & \rightarrow & I \otimes_A M & \rightarrow & (I/(I \cap m^n)) \otimes_A M \\
& & \downarrow & & \alpha & & \downarrow & & \\
m^n \otimes_A M & \rightarrow & M & \rightarrow & (A/m^n) \otimes_A M.
\end{array}
\]

with right exact rows. The cokernel of the map \( I/(I \cap m^n) \rightarrow A/m^n \) is \( A/(I + m^n) \). It has finite length by proposition 1.4.4. Thus \( \text{Tor}^A_1(A/(I + m^n), M) = 0 \) and the morphism \( \beta_n \) is injective. As a consequence, \( \ker(\alpha) \) is contained in the image of \( (I \cap m^n) \otimes_A M \).

The filtration \( m^n \) on \( A \) is \( m \)-stable. Hence by Artin-Rees lemma the filtration \( I \cap m^n \) on \( I \) is \( m \)-stable, and so the filtration on \( I \otimes_A M \) by images of \( (I \cap m^n) \otimes_A M \) is \( m \)-stable (notice that \( I \otimes_A M \) is not necessarily an \( A \)-module of finite type!).

The module \( I \otimes_A M \) has a structure of \( B \)-module via \( M \), and the images of \( (I \cap m^n) \otimes_A M \) in this module are \( B \)-submodules. Let \( J = mB \subset B \). This ideal is proper since \( A \rightarrow B \) is a local morphism. The filtration on \( I \otimes_A M \) as a \( B \)-module is \( J \)-stable. Now, Krull intersection theorem tells us that \( \ker(\alpha) = 0 \) as a submodule of zero module. \( \square \)

**Lemma 1.5.2.** Let \( A \rightarrow B \) be a local morphism of local noetherian rings, \( I \subset A \) an ideal, and \( M \) a \( B \)-module of finite type. If \( \text{Tor}^A_1(A/I, M) = 0 \) and \( M/IM \) is a flat \( A/I \)-module, then \( M \) is a flat \( A \)-module.

**Proof.** Let \( k \) be the residue field of \( A \). A short exact sequence

\[
0 \rightarrow K \rightarrow A/I \rightarrow k \rightarrow 0.
\]

yields an exact sequence

\[
\text{Tor}^A_1(A/I, M) \rightarrow \text{Tor}^A_1(k, M) \rightarrow K \otimes_A M \rightarrow A/I \otimes_A M
\]

By assumptions \( \text{Tor}^A_1(A/I, M) = 0 \). The modules \( K \) and \( A/I \) are \( A/I \)-modules, and the functor \( \otimes_A M \) restricted to such modules is isomorphic to \( \otimes_{A/I} M/IM \). The latter functor is exact, and so the arrow \( K \otimes_A M \rightarrow A/I \otimes_A M \) is injective. Hence \( \text{Tor}^A_1(k, M) = 0 \), and the local criterion of flatness finishes the proof. \( \square \)
Proposition 1.5.3. Let $A$ be a ring, $M$ a flat $A$-module. If $M/mM \neq 0$ for every $m \in \text{Specmax} A$, then $N \otimes_A M = 0$ implies $N = 0$.

**Proof.** If $m \in \text{Specmax} A$, then

$$(N \otimes_A M) \otimes_A k(m) = N/mN \otimes_{k(m)} M/mM.$$ 

Since $N \otimes_A M = 0$, we see that $N/mN = 0$ for every $m \in \text{Specmax} A$. If $N$ is of finite type, then by Nakayama $N_m = 0$ for every $m \in \text{Specmax} A$, so $N = 0$. If $N$ is not of finite type, then we take an element $x \in N$ and consider a submodule $N'$ generated by $x$. The morphism $N' \to N$ is injective, so $N' \otimes_A M \to N \otimes_A M$ is injective, and as a consequence $N' = 0$, i.e. $x = 0$. Hence, $N = 0$. \qed

Theorem 1.5.4 (Critère de platitude par fibres, cas noethérien). Let $A \to B \to C$ be local morphisms of local noetherian rings, and $M$ a $C$-module of finite type. Let $k$ be the residue field of $A$. If $M$ is nonzero, flat over $A$, and $M \otimes_A k$ is flat over $B \otimes_A k$, then $B$ is flat over $A$ and $M$ is flat over $B$.

**Proof.** Let $m$ be the maximal ideal of $A$, and $I = mB$. The natural map $m \otimes_A B \to I$ is surjective, and $(m \otimes_A B) \otimes_B C = m \otimes_A C$, so $m \otimes_A C \to I \otimes_B C$ is surjective. As a consequence, $m \otimes_A M \to I \otimes_B M$ is surjective.

The composition $m \otimes_A M \to I \otimes_B M \to M$ is injective, since $M$ is flat over $A$. Hence $m \otimes_A M \to I \otimes_B M$ is an isomorphism, and $I \otimes_B M \to M$ is injective. In particular, $\text{Tor}_1^A(B/I, M) = 0$, so $M$ is flat over $B$ by lemma 1.5.2.

Consider an exact sequence $0 \to m \to A \to k \to 0$. Tensoring with $B$ over $A$ gives us an exact sequence $0 \to \text{Tor}_1^A(k, B) \to m \otimes_A B \to I \to 0$. Tensoring the latter sequence with $M$ over $B$ yields a sequence $0 \to \text{Tor}_1^A(k, B) \otimes_B M \to m \otimes_A M \to I \otimes_B M \to 0$. The last map is an isomorphism, so $\text{Tor}_1^A(k, B) \otimes_B M = 0$.

If $m_B \subset B$ and $m_C \subset C$ are maximal ideals, then $M/m_CM$ is nonzero by Nakayama, so $M/m_B M$ is nonzero. Hence, proposition 1.5.3 applies and shows that $\text{Tor}_1^A(k, B) = 0$. It remains to apply theorem 1.5.4. \qed

1.6 Flatness in the context of schemes

**Definition 1.6.1.** Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is flat over $Y$ at $x \in X$ if the stalk $\mathcal{F}_x$ is a flat module over $\mathcal{O}_{Y,f(x)}$. We say that $f$ is flat at $x \in X$ if $\mathcal{O}_X$ is flat over $Y$ at $x$. We say that $\mathcal{F}$ is flat over $Y$ if it is flat over $Y$ at all points. We say that $f$ is flat if $\mathcal{O}_X$ is flat over $Y$.

**Proposition 1.6.2.** Flat morphisms have following properties:

(1) If $X$ and $Y$ are affine schemes and $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}$ is flat over $Y$ if and only if $\Gamma(X, \mathcal{F})$ is a flat module over $\Gamma(Y, \mathcal{O}_Y)$. 

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(2) Let $f : X \to Y$, $g : Y \to Z$ be morphisms, and $F$ a quasi-coherent sheaf. If $F$ is flat over $Y$ and $g$ is flat, then $F$ is flat over $Z$. In particular, a composition of flat morphisms is flat.

(3) Let $X \to Y$ be a morphism, $F$ a quasi-coherent sheaf, $g : Z \to Y$ a morphism, and $p : X \times_Y Z \to X$ a projection. If $F$ is flat over $Y$, then $p^*F$ is flat over $Z$. In particular, a basechange of a flat morphism is flat.

(4) An open immersion is flat.

Proof. Follows easily from what we have already done. \hfill \Box

Theorem 1.6.3. Let $S, X, Y$ be locally noetherian schemes, and $f : X \to Y$ a morphism of schemes over $S$. Let $F$ a coherent $O_X$-module. Assume that all stalks of $F$ are nonzero, $F$ is flat over $S$, and for every $s \in S$ the pullback of $F$ to $X_s$ is flat over $Y_s$. Then $F$ is flat over $Y$ and $Y$ is flat over $S$ at all points $y \in f(X)$.

Proof. Follows at once from theorem 1.5.4. \hfill \Box

Corollary 1.6.4. Let $S, X, Y$ be locally noetherian schemes. Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. If $g f$ is flat and for every $s \in S$ the pullback $X_s \to Y_s$ of $f$ is flat, then $f$ is flat, and $g$ is flat at all points $y \in f(X)$.

2 Étale morphisms

2.1 The module of Kähler differentials

Definition 2.1.1. Let $A \to B$ be a morphism of rings, and $M$ a $B$-module. An $A$-derivation $d : B \to M$ is an $A$-module morphism, which satisfies Leibnitz identity: $d(b_1 b_2) = b_2 d(b_1) + b_1 d(b_2)$ for every $b_1, b_2 \in B$.

A sum of two derivations is again an $A$-derivation, as well as a scalar multiple of a derivation by an element of $B$. Hence, $A$-derivations $B \to M$ form a $B$-module, which is denoted $\text{Der}_A(B, M)$. The association $M \mapsto \text{Der}_A(B, M)$ is a covariant functor in an evident way.

Proposition 2.1.2. Let $A \to B \to C$ be morphisms of rings. There is an induced exact sequence of functors

$$0 \to \text{Der}_B(C, -) \to \text{Der}_A(C, -) \to \text{Der}_A(B, C -)$$

The first map takes a $B$-derivation and interprets it as an $A$-derivation. The second map precomposes a derivation with the morphism $B \to C$. The symbol $B -$ denotes restriction of scalars from $C$ to $B$. 

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Proof. The first map is obviously injective. If an $A$-derivation $d: C \to M$ vanishes when restricted to $B$, then it is a $B$-derivation, so the sequence is exact at $\text{Der}_A(C, -)$. 

**Proposition 2.1.3.** Let $A \to B$ be a morphism of rings, $I \subset B$ an ideal. There is an exact sequence of functors:

$$0 \to \text{Der}_A(B/I, -) \to \text{Der}_A(B, B/I -) \to \text{Hom}_{B/I}(I/I^2, -)$$

The first map precomposes a derivation with $B \to B/I$. The second map restricts a derivation to $I$.

**Proof.** Let $M$ be a $B/I$-module. Leibnitz identity and the fact that $IM = 0$ show that every $A$-derivation $d: B \to M$ vanishes on $I^2$, and determines a morphism $d: I/I^2 \to M$ of $B/I$-modules. On the other hand, if $d|_I = 0$, then clearly $d$ comes from an $A$-derivation $B/I \to M$, hence the sequence is exact. 

**Proposition 2.1.4.** Let $A \to B$ be a morphism of rings, $s \in B$ a unit, $b \in B$ an element, and $d: B \to M$ an $A$-derivation.

$$d\left(\frac{b}{s}\right) = \frac{sdb - bds}{s^2}$$

**Proof.** From the formula $0 = d(1) = d(ss^{-1}) = sd(s^{-1}) + s^{-1}ds$ we conclude that $d\left(\frac{1}{s}\right) = -\frac{ds}{s^2}$, and then the claim follows by Leibnitz identity. 

**Proposition 2.1.5.** Let $A$ be a ring, $S \subset A$ a multiplicative system, and $A \to A_S$ a localization morphism. The functor $\text{Der}_A(A_S, -)$ is zero.

**Proof.** From the previous proposition it follows that every $A$-derivation of $A_S$ is zero.

**Proposition 2.1.6.** Let $A \to B$ be a morphism of rings, and $S \subset B$ a multiplicative system. The morphism $\text{Der}_A(B, B- ) \to \text{Der}_A(B_S, -)$ induced by $B \to B_S$ is an isomorphism.

**Proof.** Let $M$ be a $B_S$-module. We first show that the morphism in question is surjective. Let $d: B \to M$ be an $A$-derivation. It induces a derivation $D: B_S \to M$ by the rule

$$D\left(\frac{b}{s}\right) = \frac{sdb - bds}{s^2}.$$ 

Additivity and Leibnitz identity follow from trivial but lengthy calculations. Clearly, $D\left(\frac{1}{s}\right) = \frac{db}{s}$, so $D$ is an $A$-derivation which restricts to $d$ on $B$.

As for injectivity, consider the exact sequence of proposition [2.1.2] induced by $A \to B \to B_S$, and observe that $\text{Der}_B(B_S, -) = 0$. 

\[10\]
Proposition 2.1.7. Let $A$ be a ring, let $B, C$ be $A$-algebras. The morphism $\text{Der}_C(B \otimes_A C, -) \to \text{Der}_A(B, B -)$ induced by ring morphism $B \to B \otimes A C$ is an isomorphism.

Proof. Let $M$ be a module over $B \otimes A C$. An element of $\text{Der}_C(B \otimes A C, M)$ is a bilinear map $d: B \times C \to M$ which satisfies the following identities for every $a \in A, b \in B, c \in C, b_i \in B$:

\[
\begin{align*}
    d(ab, c) &= d(b, ac) = ad(b, c), \\
    d(b, c) &= (1 \otimes A c) d(b, 1), \\
    d(b_1 b_2, 1) &= (b_1 \otimes_A 1) d(b_2, 1) + (b_2 \otimes_A 1) d(b_1, 1).
\end{align*}
\]

From this description it is clear that if $d$ vanishes in $\text{Der}_A(B, M)$, then $d = 0$. Given $D \in \text{Der}_A(B, M)$ we define $d(b, c) = (1 \otimes_A c) D(b)$, which clearly satisfies the equation above, so the claim follows.

Proposition 2.1.8. Let $f: A \to B$ be a morphism of rings, $S \subset B$ a multiplicative system. The natural morphism $\text{Der}_{A^{-1} S}(B_S, -) \to \text{Der}_A(B, -)$ induced by ring morphisms $B \to B_S$ and $A \to A_{f^{-1} S}$ is an isomorphism.

Proof. The morphism in question factors as $\text{Der}_{A^{-1} S}(B_S, -) \to \text{Der}_A(B_S, -) \to \text{Der}_A(B, -)$. Since $B_S \otimes_A A_{f^{-1} S} = B_S$, the first morphism is an isomorphism by proposition 2.1.7. The second morphism is an isomorphism by proposition 2.1.6.

Theorem 2.1.9. Let $A \to B$ be a morphism of rings. The functor $\text{Der}_A(B, -)$ is representable.

Proof. Such proofs are better done on one’s own.

Let $f: X \to Y$ be a morphism of schemes. One can extend the definition of $\Omega^1$ to $X \to Y$ in two ways. First, since $\Omega^1_{B/A}$ commutes with restrictions to principal open subsets of Spec $B$ and pullbacks to principal open subsets of Spec $A$, one can pick a covering $U_i$ of $Y$ by open affines and coverings $V_{ij}$ of $f^{-1} U_i$ by open affines, then glue various $\Omega^1_{V_{ij}/U_i}$, and show that this construction does not depend on the choice of covers. The other way is, given a morphism $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces and a $\mathcal{O}_X$-module $\mathcal{F}$, define a $\mathcal{O}_X$-module of derivations $\text{Der}_{f^{-1} \mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$. One then shows that whenever $X, Y$ are schemes and the morphism $f$ is local, $\text{Der}_{f^{-1} \mathcal{O}_Y}(\mathcal{O}_X, -)$ is represented by a quasi-coherent $\mathcal{O}_X$-module, which agrees with $\Omega^1$ when $X$ and $Y$ are affine. Either way, one obtains the following theorem:

Theorem 2.1.10. To every morphism of schemes $f: X \to Y$ one can associate a quasi-coherent $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ which has following properties:
• If \( X, Y \) are affine, then \( \Omega^1_{X/Y} \) coincides with the module of Kähler differentials associated to the ring morphism \( \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X) \).

• \( \Omega^1_{X/Y} \) commutes with restrictions to opens \( U \subset X \).

• Let \( X \xrightarrow{f} S \) and \( Y \xrightarrow{g} S \) be morphism. The sheaf \( \Omega^1_{X \times_S Y/Y} \) is isomorphic to \( p^* \Omega^1_{X/S} \), where \( p: X \times_S Y \to X \) is a projection.

• If \( X \xrightarrow{f} Y \xrightarrow{g} Z \) are morphisms, then there is an exact sequence
  \[
  f^* \Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.
  \]

• If \( X \xrightarrow{f} Y \) is a morphism and \( Z \xrightarrow{g} X \) is a closed immersion with ideal sheaf \( I \), then there exists an exact sequence
  \[
  I/I^2 \to g^* \Omega^1_{X/Y} \to \Omega^1_{Z/Y} \to 0.
  \]

• If \( f: X \to Y \) is locally of finite type, then \( \Omega^1_{X/Y} \) is locally of finite type (in particular, coherent if \( X \) is locally noetherian).

2.2 Étale algebras over fields

**Proposition 2.2.1.** Let \( k \to K \) be a finite extension of fields. \( \Omega^1_{K/k} \) vanishes if and only if \( k \to K \) is separable.

**Proof.** Assume that \( k \to K \) is finite and separable. Let \( x \in K \) be a primitive element, \( f \) its minimal polynomial. Let \( M \) be a \( K \)-module, and \( d: K \to M \) a derivation.

\[
0 = d(f(x)) = f'(x)dx.
\]

Since \( K \) is separable, \( f'(x) \neq 0 \), so \( dx = 0 \) in \( M \). Since \( K \) is generated over \( k \) by powers of \( x \), we conclude that \( d = 0 \).

Assume that \( k \to K \) is inseparable and primitive. Let \( x \in K \) be a primitive element and \( f \) its minimal polynomial. Write \( K = k[T]/(f) \). Recall that every derivation \( d \in \text{Der}_k(k[T], K) \) is determined by \( d(T) \) and \( d(T) \) can be arbitrary. Set \( d(T) = x \). Then \( d \) vanishes when restricted to \( (f) \), since \( d(gf) = g(x)f'(x)dx + f(x)dg = 0 \) as \( f(x) = 0 \) and \( f'(x) = 0 \). Hence \( d \) comes from some derivation in \( \text{Der}_k(k[T]/(f), K) \) i.e. \( \text{Der}_k(K, K) \). As a consequence, \( \text{Der}_k(K, K) \neq 0 \).

Assume that \( k \to K \) is inseparable. There is a nontrivial proper subfield \( E \subset K \) such that \( E \to K \) is inseparable and primitive. Then \( \Omega^1_{K/E} \) is nonzero, since its quotient \( \Omega^1_{K/E} \) is nonzero. \( \square \)
Proposition 2.2.2. Let $k$ be an algebraically closed field, $A$ a $k$-algebra of finite type, and $m \in \text{Spec}_{\text{max}} A$. The homomorphism $m/m^2 \rightarrow \Omega_{A/k}^1 \otimes_A k(m)$ is an isomorphism.

Proof. We need to prove that the natural restriction map
\[ \text{Der}_k(A, M) \rightarrow \text{Hom}_{A/m}(m/m^2, M) \]
is an isomorphism for every $A/m$-module $M$.

By Hilbert’s Nullstellensatz the composition $k \rightarrow A \rightarrow A/m$ is an isomorphism. In particular, $\text{Der}_k(A/m, -) = 0$, so that the natural map in question is injective.

Let $f : m/m^2 \rightarrow M$ be a morphism of $A/m$-modules. We define a map $d : A \rightarrow M$ by sending an element $a \in A$ to $f(a - a(m))$, where $a(m)$ is the image of $a$ modulo $A$. If $a_1, a_2 \in A$, then
\begin{align*}
    a_1a_2 - a_1(m)a_2(m) &= (a_1 - a_1(m))(a_2 - a_2(m)) + a_2(m)(a_1 - a_1(m)) + a_1(m)(a_2 - a_2(m)).
\end{align*}
Also, $a_i = a_i(m)$ in $A/m$, so that $d(a_1a_2) = a_2d(a_1) + a_1d(a_2)$. Clearly, $d$ vanishes on elements of $k$, so it is a derivation. \qed

Definition 2.2.3. Let $k$ be a field. A $k$-algebra $A$ is called étale if it is a finite cartesian product of finite separable extensions of $k$.

Theorem 2.2.4. Let $k$ be a field. A $k$-algebra of finite type $A$ is étale if and only if $\Omega_{A/k}^1 = 0$.

Proof. Let $A$ be a $k$-algebra of finite type such that $\Omega_{A/k}^1 = 0$. Let us first assume that $k$ is algebraically closed. By virtue of proposition 2.2.2 we then know that $m/m^2 = 0$ for every maximal ideal $m$ of $A$. Localizing at $m$ and applying Nakayama lemma we conclude that $A_m$ is a field, the kernel of the localization morphism $A \rightarrow A_m$ is $m$, and $A_m = A/m$. By Nullstellensatz, $A/m \cong k$.

Let $p \in \text{Spec} A$ be a prime, and let $m$ be a maximal ideal containing it. Let $a \in m$. Since $a$ vanishes in $A_m$, there exists $s \notin m$ such that $sa = 0$ in $A$. In particular, $sa \in p$, so $a \in p$. Hence each prime of $A$ is maximal.

The algebra $A$ is noetherian, so that the set of its minimal primes is finite. But all primes are maximal, so $\text{Spec}_{\text{max}} A$ is finite. Now, consider a morphism
\[ A \rightarrow \prod_{m \in \text{Spec}_{\text{max}} A} A/m \] (1)

By Chinese remainder theorem it is surjective. But $A/m = A_m$, so that the kernel of this morphism consists of elements which vanish in all localizations.
of $A$ at maximal ideals, i.e. the kernel is zero. Hence, this morphism is an isomorphism. In particular, dim$_k A$ is finite.

Now, let $k$ be arbitrary, and $\bar{k}$ its algebraic closure. Let $A_{\bar{k}} = A \otimes_k \bar{k}$. Since dim$_{\bar{k}} A_{\bar{k}}$ is finite, dim$_k A$ is finite too. Let $p \in \text{Spec} A$ be a prime. The $k$-algebra $A/p$ is finite-dimensional and has no zero divisors, hence it is a field. So Spec $A = \text{Specmax} A$, and Specmax $A$ is finite.

We consider a morphism as in (1). Its kernel is the nilradical of $A$. If $a \in A$ is nilpotent, then its image in $A_{\bar{k}}$ is nilpotent too, hence zero. But $A \to A_{\bar{k}}$ is injective, so that the kernel of (1) is zero. Now, proposition 2.2.1 finishes the proof.\]

2.3 Unramified morphisms

**Definition 2.3.1.** Let $f: X \to Y$ be a morphism of schemes. We say that $f$ is unramified if $f$ is locally of finite type and $\Omega^1_{X/Y} = 0$.

**Proposition 2.3.2.** Unramified morphisms have following properties:

(1) If $f: X \to Y$ and $g: Y \to Z$ are unramified, then $gf$ is unramified.

(2) If $f: X \to Y$ and $g: Y \to Z$ are such that $gf$ is unramified, then $f$ is unramified.

(3) If $f: X \to S$ is unramified, and $g: Y \to S$ is a morphism, then the pullback $X \times_S Y \to Y$ of $f$ is unramified.

(4) Open immersions are unramified.

**Proof.** (1) The composition $gf$ is locally of finite type. The exact sequence

$$g^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

implies that $\Omega^1_{X/Z} = 0$.

(2) The exact sequence above shows that $\Omega^1_{X/Y} = 0$. The fact that $f$ is locally of finite type is left as an exercise (see [2] tag 01T8).

(3) Follows from proposition 2.1.7

(4) Follows from proposition 2.1.5 \]

**Proposition 2.3.3.** Let $f: X \to Y$ be a morphism locally of finite type. It is unramified if and only if for each $y \in Y$ the fiber $X_y \to y$ is unramified.
Proof. If $\Omega^1_{X/Y} = 0$, then clearly each fiber is unramified. Conversely, if $X_y \to y$ is unramified, then the fiber of $\Omega^1_{X/Y}$ at each point $x \in X$ is zero, as an inclusion of a point $x \in X$ factors through $X_{f(y)} \to X$. Since $\Omega^1_{X/Y}$ is locally of finite type, Nakayama lemma shows that $\Omega^1_{X/Y} = 0$.

**Proposition 2.3.4.** Let $X \to \text{Spec } k$ be a scheme over a field. It is unramified if and only if $X$ is discrete as a topological space, and for every $x \in X$ the field extension $k \to k(x)$ is finite separable.

Proof. Assume that $X \to \text{Spec } k$ is unramified. Let $x \in X$ and $U \subset X$ be an affine open neighbourhood of $x$ which is of finite type over $\text{Spec } k$. By theorem 2.2.4 we conclude that $U$ is a spectrum of an étale algebra over $k$. In particular, $U$ is discrete. Hence $X$ is discrete.

Assuming the converse, take $x \in X$ and $U \subset X$ an affine open neighbourhood of $x$. Since $X$ is discrete, $U$ is discrete too, and as $U$ is quasi-compact, we conclude that $U$ is finite as a topological space. Hence $U$ is a spectrum of an étale algebra over $k$, and so $\Omega^1_{X/k}|_U = 0$. As a consequence, $\Omega^1_{X/k} = 0$. Since $U$ is a spectrum of an algebra of finite type over $k$, we conclude that $X \to \text{Spec } k$ is locally of finite type.

**Proposition 2.3.5.** Let $X, Y$ be schemes and $f : X \to Y$ a morphism locally of finite type. The fiber of $\Omega^1_{X/Y}$ at $x$ is zero if and only if the residue field extension $k(f(x)) \to k(x)$ is finite separable, and $m_{Y,f(y)}O_{X,x} = m_{X,x}$.

Proof. We immediately reduce to the case when $X = \text{Spec } B$ and $Y = \text{Spec } A$ are affine, and $f$ is of finite type. Let $q \in \text{Spec } B$ and $p = f(q)$.

Assume that $\Omega^1_{B/A} \otimes_B k(q) = 0$. Since $\Omega^1_{B/A}$ is of finite type, Nakayama lemma implies that $(\Omega^1_{B/A})_q = 0$. Hence replacing $B$ by its localization at some element not contained in $q$ we may assume that $\Omega^1_{B/A} = 0$. As a consequence, $\Omega^1_{B_p/A_p} = 0$.

Consider a ring $B \otimes_A k(p)$. Since $\Omega^1_{B \otimes_A k(p)/k(p)} = 0$ and $B$ is of finite type over $A$, theorem 2.2.4 shows that $B \otimes_A k(p)$ is a finite étale algebra over $k(p)$.

The morphism $A_p \to k(p)$ is surjective, so $B_q \to B_q \otimes_{A_p} k(p)$ is surjective. On the other hand

$$B \otimes_A k(p) = B \otimes_A (A_p \otimes_{A_q} k(p)) = B_p \otimes_{A_p} k(p),$$

so $B_q \otimes_{A_p} k(p)$ is a localization of a finite étale algebra over $k(p)$, hence is itself such an algebra.

The morphism $\text{Spec}(B_q \otimes_{A_p} k(p)) \to \text{Spec } B_q$ is a closed immersion. In particular, it is injective and sends closed points to closed points. As $B_q$ has only one maximal ideal, we conclude that $B_q \otimes_{A_p} k(p)$ also has unique maximal ideal,
which forces it to be a finite separable field extension of \( k(p) \). On the other hand \( B_q \otimes_{A_p} k(p) = B_q/pB_q \), so that \( pB_q = qB_q \).

Now, assume that \( pB_q = qB_q \) is maximal, and that \( k(q) \) is a finite separable extension of \( k(p) \). Our assumptions imply that \( B_q/pB_q = B_q \otimes_{A_p} k(p) = k(q) \). Hence \( \Omega^1_{B/A} \otimes_B k(q) = \Omega^1_{B_q/A_p} \otimes_{B_q} k(q) = \Omega^1_{k(q)/k(p)} = 0. \)

### 2.4 Étale morphisms

**Definition 2.4.1.** Let \( f : X \to Y \) be a morphism of schemes. We say that \( f \) is étale if it is unramified and flat.

**Proposition 2.4.2.** Étale morphisms have following properties:

1. If \( f : X \to Y \) and \( g : Y \to Z \) are étale, then \( gf \) is étale.

2. If \( f : X \to S \) is étale, and \( g : Y \to S \) is a morphism, then the pullback \( X \times_S Y \to Y \) of \( f \) is étale.

3. Open immersions are étale.

4. If a morphism \( f : X \to Y \) of schemes is locally of finite type, flat, and every fiber \( X_y \to y \) is unramified, then \( f \) is étale.

**Proof.** Everything follows at once from corresponding properties of flat and unramified morphisms.

**Proposition 2.4.3.** Let \( f : X \to Y \) and \( g : Y \to S \) be morphisms of schemes. If \( gf \) is étale and \( g \) is unramified, then \( f \) is étale. If in addition \( f \) is surjective, then \( g \) is étale.

**Proof.** Follows from corollary [1.6.4] because each fiber \( Y_s \) is a disjoint union of spectra of fields.

### References


2. A.J. de Jong et al., *Stacks Project*