THE PRO-ÉTALE COHOMOLOGY OF $\underline{\mathbb{Z}_\ell}$

MAXIM MORNEV

To facilitate reading of [1] we go into more details than what is actually needed to compute the cohomology of $\underline{\mathbb{Z}_\ell}$. Some degree of familiarity with derived categories is assumed. In the following when we mention derived categories we silently assume that they exist.[1]

1. How it fails in the étale setting

Let X be a scheme, and ℓ a prime not divisible in X. Consider the sheaf $\underline{\mathbb{Z}_\ell} = \lim_n \underline{\mathbb{Z}/\ell^n}$ on $X_{\text{ét}}$. If X is a smooth connected projective curve of geometric genus g over an algebraically closed field then

$$H^k(X_{\text{ét}}, \underline{\mathbb{Z}_\ell}) = \begin{cases} \mathbb{Z}_\ell, & k = 0, \\ 0, & k = 1, \\ (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}, & k = 2, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell, & k = 3, \\ 0, & k > 3 \end{cases}$$

See [3], chapter I, paragraph 12.

2. Products in derived categories

Proposition 2.1. *Let $\mathcal{A}$ be an abelian category, $\{K_n\}_{n\in\mathbb{N}}$ a family of complexes in $\mathcal{A}$. If countable products are exact in $\mathcal{A}$ then the termwise product $\prod_n K_n$ represents the product of K_n in $D(\mathcal{A})$.*

Proof. Let $L \in D(\mathcal{A})$, and let $\{\alpha_n : L \to K_n\}_{n\in\mathbb{N}}$ be a family of morphisms in $D(\mathcal{A})$. So for each $n \in \mathbb{N}$ there is a complex M_n, a morphism of complexes $f_n : L \to M_n$, and a quasi-isomorphism $s_n : K_n \to M_n$. By exactness of countable products $\prod_n s_n$ is a quasi-isomorphism, so that we get a morphism $\alpha : L \to \prod_n K_n$ whose compositions with projections $\pi_n : \prod_n K_n \to K_n$ are equal to α_n. Another application of exactness of products shows that this morphism is independent of the choice of fractions representing α_n's. So it only remains to show unicity.

Assume that we are given another complex L', a morphism of complexes $g : L \to L'$, and a quasi-isomorphism $s : \prod_n K_n \to L'$. The fact that the composition of this fraction with projection $\pi_n : \prod_n K_n \to K_n$ is equal to α_n in $D(\mathcal{A})$ means that there exists a morphism $g_n : L' \to L'_n$, and a quasi-isomorphism $t_n : K_n \to L'_n$ of

[1] Unfortunately the meaning of the word "exists" depends on the choice of foundations.

complexes such that $g_n s = t_n \pi_n$, and the fraction defined by $g_n g$, and t_n represents α_n.

Now, $\prod_n t_n \colon \prod_n K_n \to \prod_n L'_n$ is a quasi-isomorphism by exactness of products. Let $g' \colon L' \to \prod_n L'_n$ be the morphism of complexes defined by the family $\{g_n \colon L' \to L'_n\}_{n\in\mathbb{N}}$. The fraction defined by $\prod_n t_n$, and $g'g$ represents α because our construction of a morphism $L \to \prod_n K_n$ in $D(\mathcal{A})$ is independent of the choice of representing fractions.

At the same time this fraction is equivalent to the one defined by $g \colon L \to L'$, and $s \colon \prod_n K_n \to L'$. To see this notice that g' is a quasi-isomorphism. Indeed the diagram

$$\begin{array}{ccc} L' & \xrightarrow{g'} & \prod_n L'_n \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle \prod_n t_n} \\ \prod_n K_n & \xrightarrow{1} & \prod_n K_n \end{array}$$

commutes up to homotopy. □

3. The shift map

Let $\mathcal{C}$ be a category, $F \colon \mathbb{N}^\circ \to \mathcal{C}$ a diagram. Suppose that the product $\prod_{n\in\mathbb{N}} F_n$ exists. Consider the unique morphism $\tau \colon \prod_n F_n \to \prod_n F_n$ such that the diagrams

$$\begin{array}{ccc} F_{n+1} & \xrightarrow{F(n+1\leftarrow n)} & F_n \\ {\scriptstyle \pi_{n+1}}\uparrow & & \uparrow{\scriptstyle \pi_n} \\ \prod\limits_n F_n & \xrightarrow{\tau} & \prod\limits_n F_n \end{array}$$

commute for all n.

Proposition 3.1. *The equalizer of the diagram*

$$\prod_n F_n \underset{\tau}{\overset{1}{\rightrightarrows}} \prod_n F_n$$

is $\lim F$.

Proof. Omitted. □

4. Derived limits

Let $\mathcal{A}$ be an abelian category with exact countable products. Let $\mathcal{A}(\mathbb{N})$ be the category of functors $\mathbb{N}^\circ \to \mathcal{A}$. It is also an abelian category.

Let $\pi_n \colon \mathcal{A}(\mathbb{N}) \to \mathcal{A}$ be the functor which sends a diagram $A \colon \mathbb{N}^\circ \to \mathcal{A}$ to $A(n)$, and let $\pi_{n+1} \to \pi_n$ be the natural transformation induced by the arrow $n \to (n+1)$.

The functors π_n pass to the level of categories of complexes. Moreover they descend to the level of derived categories as exact functors.

Define a functor $\mathrm{Rlim}\colon \mathrm{Comp}(\mathcal{A}(\mathbb{N})) \to \mathrm{Comp}(\mathcal{A})$ as:

$$\mathrm{Rlim}\, A = \mathrm{cone}\left(\prod_n \pi_n(A) \xrightarrow{1-\tau} \prod_n \pi_n(A)\right)[-1].$$

Proposition 4.1. (1) Rlim *descends to the level of homotopy categories.*

(2) Rlim *is exact on the level of homotopy categories.*

(3) Rlim *sends acyclic complexes to acyclic complexes.*

Proof. (1), (2) Omitted. Hint: decompose Rlim into functors $\mathrm{Comp}(\mathcal{A}(\mathbb{N})) \to \mathrm{Comp}(\mathcal{A}(\to)) \to \mathrm{Comp}(\mathcal{A})$.

(3) Follows from exactness of countable products. □

We therefore obtain a functor $\mathrm{Rlim}\colon D(\mathcal{A}(\mathbb{N})) \to D(\mathcal{A})$. As its name suggests it is the derived functor of $\lim\colon \mathcal{A}(\mathbb{N}) \to \mathcal{A}$. A proof can be found in [8], appendix A.3. However we will not need this fact.

We will use notation $\mathrm{R}^i\mathrm{lim}$ for i-th cohomology of Rlim.

Proposition 4.2. *Let* $A\colon \mathbb{N}^\circ \to \mathcal{A}$ *be a diagram.* $\mathrm{R}^0\mathrm{lim}\, A = \lim A_n$, *and* $\mathrm{R}^i\mathrm{lim}\, A = 0$ *whenever* $i \notin \{0, 1\}$.

Proof. Omitted. □

Proposition 4.3. *Let* $A \in D(\mathcal{A}(\mathbb{N}))$. *There is a canonical distinguished triangle*

$$\mathrm{Rlim}\, A \to \prod_n \pi_n(A) \xrightarrow{1-\tau} \prod_n \pi_n(A) \to \mathrm{Rlim}\, A[1] \tag{1}$$

in $D(\mathcal{A})$.

Proof. Follows since products in $D(\mathcal{A})$ are termwise. □

5. Mittag-Leffler criterion

Definition 5.1. We say that a diagram $A\colon \mathbb{N}^\circ \to \mathcal{A}$ satisfies the Mittag-Leffler condition if for every n and every $m \geqslant n$ large enough the image of A_m in A_n is independent of m.

Now we switch to the case $\mathcal{A} = \mathrm{Ab}$.

Proposition 5.2 (Mittag-Leffler criterion for Ab)**.** *Let* $A\colon \mathbb{N}^\circ \to \mathrm{Ab}$ *be a diagram. If* A *satisfies Mittag-Leffler condition then* $\mathrm{R}^1\mathrm{lim}\, A = 0$.

Proof. Taken from [10], section 3.5.

(1) Let $A'_n \subset A_n$ denote the stable image of A_m, $m \geqslant n$. Note that the modules A'_n form a subdiagram of A, and the transition maps $A'_{n+1} \to A'_n$ are surjections. Therefore A/A' is a diagram. This diagram has the property that for every n the image of $(A/A')_m$ in $(A/A')_n$ is zero for m large enough.

Since Rlim is an exact functor between triangulated categories the short exact sequence

$$0 \to A' \to A \to A/A' \to 0$$

induces a cohomology exact sequence

$$\mathrm{R}^1\mathrm{lim}\, A' \to \mathrm{R}^1\mathrm{lim}\, A \to \mathrm{R}^1\mathrm{lim}(A/A') \to 0.$$

Therefore to show that $\mathrm{R}^1\mathrm{lim}\, A = 0$ it is enough to show that $\mathrm{R}^1\mathrm{lim}\, A' = 0$, and $\mathrm{R}^1\mathrm{lim}(A/A') = 0$.

(2) Assume that for every n, and $m \geqslant n$ large enough the image of A_m in A_n is zero. We will show that the map

$$\prod_n A_n \xrightarrow{1-\tau} \prod_n A_n$$

is onto. Let $(a_n) \in \prod_n A_n$. Fix $n \in \mathbb{N}$, and $m \geqslant n$ such that $A_m \to A_n$ is zero. Set $\alpha_n = a_n + \overline{a}_{n+1} + \ldots + \overline{a}_{m-1}$ where $\overline{a}_i$ denotes the image of a_i in A_n via the composite map $A_i \to A_n$. The result is straighforward to check.

(3) Assume that all the maps $A_{n+1} \to A_n$ are surjective. Given $(a_n) \in \prod_n A_n$ let $\alpha_0 \in A_0$ be arbitrary, and let α_{n+1} be a lift of $\alpha_n - a_n$ for $n > 1$. □

Thanks to Amnon Neeman and Pierre Deligne [7] it is now known that there are abelian categories $\mathcal{A}$ with exact products in which the Mittag-Leffler criterion does not hold. More concretely, there are diagrams $A\colon \mathbb{N}^\circ \to \mathcal{A}$ with epimorphic transition maps, and $\mathrm{R}^1\mathrm{lim}\, A \neq 0$. Note, however that Jan Erik Roos [9] and Ofer Gabber proved the following theorem (theorem 3.1 in [9]):

Theorem. *Let $\mathcal{A}$ be an abelian category with AB3, AB4*, and a generator. If $A\colon \mathbb{N}^\circ \to \mathcal{A}$ is a diagram satisfying Mittag-Leffler condition then* $\mathrm{R}^1\mathrm{lim}\, A = 0$.

6. Homotopy limits

Let $\mathcal{T}$ be a triangulated category, $F\colon \mathbb{N}^\circ \to \mathcal{T}$ a diagram. Suppose that the product $\prod_{n\in\mathbb{N}} F_n$ exists. We define $\mathrm{holim}\, F$ as an object which fits into a distinguished triangle

$$\mathrm{holim}\, F \to \prod_n F_n \xrightarrow{1-\tau} \prod_n F_n \to \mathrm{holim}\, F[1]. \tag{2}$$

In general $\mathrm{holim}\, F$ is defined only up to a non-unique isomorphism.

Example 6.1. A homotopy limit is not expected to represent the limit of F in $\mathcal{T}$. Here is a simple example. Take $\mathcal{T} = D(\mathbb{Z})$ the derived category of $\mathbb{Z}$-modules. Let ℓ be a prime, and F the diagram

$$\to \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \xrightarrow{\ell} \mathbb{Z}.$$

in Ab. Let $K = \mathrm{Rlim}\, F \in D(\mathbb{Z})$. The object K is a homotopy limit of the diagram F viewed as a diagram in $D(\mathbb{Z})$.

The diagram F sits in a short exact sequence of diagrams

$$\begin{array}{ccccccccc}
 & & \downarrow \ell & & \downarrow 1 & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ell^3} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^3 & \longrightarrow & 0 \\
 & & \downarrow \ell & & \downarrow 1 & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ell^2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell^2 & \longrightarrow & 0 \\
 & & \downarrow \ell & & \downarrow 1 & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ell} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\ell & \longrightarrow & 0
\end{array}$$

Applying Rlim to this short exact sequence and taking cohomology we obtain an exact sequence

$$0 \to H^0(K) \to \mathbb{Z} \to \mathbb{Z}_\ell \to H^1(K) \to 0$$

since $\mathrm{R}^1\mathrm{lim}$ of the diagram $\to \mathbb{Z} \xrightarrow{1} \mathbb{Z}$ vanishes. Clearly $H^0(K) = \lim F = 0$, so $H^1(K) = \mathbb{Z}_\ell/\mathbb{Z} \neq 0$, and K is isomorphic in $D(\mathbb{Z})$ to $\mathbb{Z}_\ell/\mathbb{Z}$ placed in degree 1.

Assume that K represents $\lim F$ in $D(\mathbb{Z})$. Then for every $i \in \mathbb{Z}_{\geqslant 1}$ we must have

$$\mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{Z}[-i], K) = \lim_n \mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{Z}[-i], \mathbb{Z}[0]) = 0$$

since $\mathbb{Z}$ is projective. On the other hand, $K \cong \mathbb{Z}_\ell/\mathbb{Z}[-1]$ so

$$\mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{Z}[-1], K) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}_\ell/\mathbb{Z}) \neq 0,$$

a contradiction. □

Proposition 6.2. *Let $F\colon \mathbb{N}^\circ \to \mathcal{T}$ be a diagram. There is a short exact sequence*

$$0 \to \mathrm{R}^1\mathrm{lim}\, \mathrm{Hom}_{\mathcal{T}}(-[1], F_n) \to \mathrm{Hom}_{\mathcal{T}}(-, \mathrm{holim}\, F) \to \lim \mathrm{Hom}_{\mathcal{T}}(-, F_n) \to 0.$$

Proof. The distinguished triangle 2 induces an exact sequence

$$\begin{array}{ccccc}
\prod_n \mathrm{Hom}(-[1], F_n) & & & & \\
\downarrow{\scriptstyle 1-\tau} & & & & \\
\prod_n \mathrm{Hom}(-[1], F_n) & \longrightarrow & \mathrm{Hom}(-, \mathrm{holim}\, F) & \longrightarrow & \prod_n \mathrm{Hom}(-, F_n) \\
 & & & & \downarrow{\scriptstyle 1-\tau} \\
 & & & & \prod_n \mathrm{Hom}(-, F_n)
\end{array}$$

from which the statement is clear. □

Proposition 6.3. *Let $\mathcal{A}$ be an abelian category. Assume that countable products exist in $\mathcal{A}$, and are exact. Let $F\colon \mathbb{N}^\circ \to D(\mathcal{A})$ be a diagram. For every $i \in \mathbb{Z}$ there is a short exact sequence*

$$0 \to \mathrm{R}^1\lim_n H^{i-1}(F_n) \to H^i(\operatorname{holim} F) \to \lim_n H^i(F_n) \to 0.$$

Proof. Follows from the cohomology long exact sequence. □

Proposition 6.4. *Let $\mathcal{T}$, $\mathcal{S}$ be triangulated categories with countable products, and $F\colon \mathcal{T} \to \mathcal{S}$ an exact functor. Let $K\colon \mathbb{N}^\circ \to \mathcal{T}$ be a diagram. If F commutes with countable products then $F(\operatorname{holim} K) \cong \operatorname{holim}(F \circ K)$.*

Proof. Follows from applying F to the distinguished triangle 2. □

7. Limits in $\mathrm{Ab}(X_{\text{proét}})$

Let X be a scheme, and $X_{\text{proét}}$ the small pro-étale site over X.

One of the main properties of $X_{\text{proét}}$ is that it is locally weakly contractible. As a consequence the category $\mathrm{Ab}(X_{\text{proét}})$ has several nice homological properties. The reason is that weakly contractible objects behave at the same time as if they are opens (taking sections commutes with limits), and as if they are points (taking sections is exact). So questions about limits in $\mathrm{Ab}(X_{\text{proét}})$ can be translated to questions about limits in Ab.

Proposition 7.1. *Arbitrary products in $\mathrm{Ab}(X_{\text{proét}})$ are exact.*

Proof. Let $\{\mathcal{F}_i \to \mathcal{G}_i\}_{i\in I}$ be a family of epimorphisms of abelian sheaves in $X_{\text{proét}}$, and let $U \in X_{\text{proét}}$ be a w-contractible object. As it is true for every object of $X_{\text{proét}}$ taking sections over U commutes with limits. Hence we only need to check that the morphism

$$\prod_i \mathcal{F}_i(U) \to \prod_i \mathcal{G}_i(U)$$

is onto. This follows since products in Ab are exact. □

The fact that $X_{\text{proét}}$ is locally weakly contractible implies that the Mittag-Leffler criterion is valid in $\mathrm{Ab}(X_{\text{proét}})$.

Lemma 7.2. *Let $\mathcal{F}\colon \mathbb{N}^\circ \to \mathrm{Ab}(X_{\text{proét}})$ be a diagram of abelian sheaves. If $\mathcal{F}$ satisfies the Mittag-Leffler condition then $\mathrm{R}^1\lim \mathcal{F} = 0$.*

Proof. Let $U \in X_{\text{proét}}$ be w-contractible. Since taking sections over U is exact the system $\mathcal{F}(U)\colon \mathbb{N}^\circ \to \mathrm{Ab}$ satisfies the Mittag-Leffler condition. Hence the morphism

$$\prod_{n\in\mathbb{N}} \mathcal{F}_n(U) \xrightarrow{1-\tau} \prod_{n\in\mathbb{N}} \mathcal{F}_n(U)$$

is an epimorphism. Since it holds for every w-contractible U, we are done. □

Essentially the same proof shows that $\mathrm{Sh}(X_{\text{proét}})$ is *replete*, i.e. if $\mathcal{F}\colon \mathbb{N}^\circ \to \mathrm{Sh}(X_{\text{proét}})$ is a diagram with epimorphic transition maps then $\lim \mathcal{F} \to \mathcal{F}_n$ is an epimorphism for every $n \in \mathbb{N}$.

8. The sheaf $\underline{\mathbb{Z}}_l$

Let X be a scheme, and ℓ a prime. We work with the small pro-étale site of X.

Definition 8.1. Define presheaves $\underline{\mathbb{Z}/\ell^n}$, and $\underline{\mathbb{Z}}_\ell$ on $X_{\text{proét}}$ as

$$\underline{\mathbb{Z}/\ell^n}(S) = \mathrm{Hom}_{\mathrm{Top}}(S, \mathbb{Z}/\ell^n),$$
$$\underline{\mathbb{Z}}_\ell(S) = \mathrm{Hom}_{\mathrm{Top}}(S, \mathbb{Z}_\ell)$$

($\mathbb{Z}/\ell^n$ has discrete topology).

If U is connected then $\underline{\mathbb{Z}}_\ell(U) = \mathbb{Z}_\ell$ because $\mathbb{Z}_\ell$ is totally disconnected.

In the previous talk Jinbi explained that presheaves so defined are sheaves ([5], proposition 2.8, or [1], lemma 4.2.12). In any case we will see in a moment that they are representable, so are sheaves by faithfully flat descent.

Proposition 8.2. *There is an isomorphism*

$$\underline{\mathbb{Z}}_\ell = \lim_n \underline{\mathbb{Z}/\ell^n},$$

where the transition maps are induced by reduction $\mathbb{Z}/\ell^{n+1} \to \mathbb{Z}/\ell^n$. Moreover the transition maps are epimorphisms of presheaves.

Proof. $\mathbb{Z}_\ell$ is the limit of $\mathbb{Z}/\ell^n$ in the category of topological spaces. Moreover, the reduction map $\mathbb{Z}/\ell^{n+1} \to \mathbb{Z}/\ell^n$ is a split surjection as a map of topological spaces. □

For X a scheme and S a set we denote $X \otimes S$ the X-scheme which is the disjoint union of S copies of X.

Proposition 8.3. *The presheaf $\underline{\mathbb{Z}/\ell^n}$ is represented by the X-scheme $X \otimes \mathbb{Z}/\ell^n$.*

Proof. Omitted. □

Proposition 8.4. *The limit $X_\ell := \lim_n(X \otimes \mathbb{Z}/\ell^n)$ exists in Sch_X, represents $\underline{\mathbb{Z}}_\ell$, and is weakly étale over X.*

Proof. The limit exists in the category of schemes since the transition maps are affine (Stacks 01YX). Such a limit is also a limit in the category of schemes over X. The limit represents $\underline{\mathbb{Z}}_\ell$ because $\underline{\mathbb{Z}}_\ell$ is the limit of $\underline{\mathbb{Z}/\ell^n}$. Finally X_ℓ is weakly étale over X since it is a limit of schemes étale over X (Stacks 094S + Stacks 092N). □

If the base X is qcqs then on the level of topological spaces X_ℓ is $X \times \mathbb{Z}_\ell$ (Stacks 01YY). As a consequence X_ℓ is not locally of finite type over X. Indeed, if X is the spectrum of a field, then all points of X_ℓ are closed, and there are uncountably many of them in every neighbourhood of every point of X_ℓ, so it can not be locally noetherian. It remains to notice that formation of X_ℓ commutes with base change.

The sheaf $\underline{\mathbb{Z}}_\ell$ is not the constant sheaf with value $\mathbb{Z}_\ell$. Indeed, the latter sheaf is represented by $X \otimes \mathbb{Z}_\ell$ which is weakly étale over X but not affine. On the other hand X_ℓ is affine over X.

However the restriction of $\underline{\mathbb{Z}_\ell}$ to the small étale site $X_{\text{ét}}$ is precisely the usual étale sheaf $\underline{\mathbb{Z}_\ell}$.

9. Generalities on RΓ

An abelian category $\mathcal{A}$ is called Grothendieck if it has coproducts, filtered colimits are exact, and if it has a generator. A generator is an object U such that for every nonzero morphism $f\colon A \to B$ there exists a morphism $g\colon U \to A$ with $fg \neq 0$.

Grothendieck abelian categories have particularly nice properties. Every object of a Grothendieck category has a set of subobjects, the category has enough injectives, every complex has a K-injective (homotopy injective) resolution, and the collection of quasi-isomorphisms is locally small. So the derived category exists, as well as arbitrary right derived functors.

The category of abelian sheaves on a site is always Grothendieck. Existence and exactness of filtered colimits is Stacks 03CO, and existence of a generator follows from existence of extension by zero functor (Stacks 04BE). As a consequence for every scheme X the derived category $D(X_{\text{proét}})$ of $\mathrm{Ab}(X_{\text{proét}})$ exists, as well as the right derived functor $\mathrm{R}\Gamma(X_{\text{proét}}, -)\colon D(X_{\text{proét}}) \to D(\mathbb{Z})$ of the global sections functor $\Gamma(X_{\text{proét}}, -)$. Moreover $\mathrm{R}\Gamma(X_{\text{proét}}, -)$ is computed by $\Gamma(X_{\text{proét}}, -)$ on K-injective complexes.

Proposition 9.1. *For $K \in D(\mathbb{Z})$ let $\underline{K}$ denote the corresponding complex of constant pro-étale sheaves. There is a natural isomorphism*

$$\mathrm{RHom}(\underline{K}, L) = \mathrm{RHom}(K, \mathrm{R}\Gamma(X_{pro\acute{e}t}, L)).$$

In particular $\mathrm{R}\Gamma(X, -)$ *is right adjoint.*

Proof. We temporarily omit the subscript $_{\text{proét}}$ for brevity. By adjunction of constant sheaf functor and $\Gamma(X, -)$ we have a natural isomorphism

$$\mathrm{Hom}^\bullet(K, \Gamma(X, L)) = \mathrm{Hom}^\bullet(\underline{K}, L).$$

If K is acyclic then $\underline{K}$ is also acyclic by exactness of the constant sheaf functor. Thus if L is K-injective then the Hom complex on the right is acyclic. Hence $\Gamma(X, L)$ is K-injective. As a consequence

$$\mathrm{RHom}(K, \mathrm{R}\Gamma(X, L)) = \mathrm{Hom}^\bullet(K, \Gamma(X, L)) = \mathrm{Hom}^\bullet(\underline{K}, L) = \mathrm{RHom}(\underline{K}, L)$$

if L is K-injective. □

10. The cohomology

Proposition 10.1. $\underline{\mathbb{Z}_\ell} \cong \mathrm{Rlim}_n \underline{\mathbb{Z}/\ell^n}$ *in* $D(X_{pro\acute{e}t})$.

Proof. Follows from the Mittag-Leffler criterion. □

Proposition 10.2. *For every $i \in \mathbb{Z}$ there are short exact sequences*

$$0 \to \mathrm{R}^1\lim_n H^{i-1}(X_{pro\acute{e}t}, \underline{\mathbb{Z}/\ell^n}) \to H^i(X_{pro\acute{e}t}, \underline{\mathbb{Z}_\ell}) \to \lim_n H^i(X_{pro\acute{e}t}, \underline{\mathbb{Z}/\ell^n}) \to 0.$$

Proof. Applying RΓ to the triangle (1), and using the fact that RΓ commutes with products we conclude that

$$\mathrm{R}\Gamma(X_{\text{proét}}, \underline{\mathbb{Z}_\ell}) \cong \operatorname*{holim}_n \mathrm{R}\Gamma(X_{\text{proét}}, \underline{\mathbb{Z}/\ell^n}).$$

Now the short exact sequences follow from proposition 6.3. □

Remark 10.3. One can prove that

$$\lim_n H^i(X_{\text{proét}}, \underline{\mathbb{Z}_\ell})/\ell^n = \lim_n H^i(X_{\text{proét}}, \underline{\mathbb{Z}/\ell^n}),$$

and that $\mathrm{R}^1\lim_n H^{i-1}(X_{\text{proét}}, \underline{\mathbb{Z}/\ell^n})$ vanishes if and only if $H^i(X_{\text{proét}}, \underline{\mathbb{Z}_\ell})$ is ℓ-adically separated.

Proposition 10.4. *For every n there are natural isomorphisms*

$$\mathrm{R}\Gamma(X_{pro\acute{e}t}, \underline{\mathbb{Z}/\ell^n}) \cong \mathrm{R}\Gamma(X_{\acute{e}t}, \underline{\mathbb{Z}/\ell^n})$$

in $D(\mathbb{Z})$.

Proof. The inclusion of $X_{\text{ét}}$ to $X_{\text{proét}}$ gives a morphism of sites $\varepsilon\colon X_{\text{proét}} \to X_{\text{ét}}$. The sheaf $\underline{\mathbb{Z}/\ell^n}$ on $X_{\text{ét}}$ is represented by $X \otimes \mathbb{Z}/\ell^n$. Hence its pullback by ε is the pro-étale $\underline{\mathbb{Z}/\ell^n}$. The result now follows from Stacks 099W. □

Remark 10.5. In general if $K \in D^+(X_{\text{ét}})$ then $\mathrm{R}\Gamma(X_{\text{ét}}, K) = \mathrm{R}\Gamma(X_{\text{proét}}, \varepsilon^{-1}K)$. This formula does not hold for unbounded complexes, and should not hold. See [1], remark 5.1.8, and section 3.3 for details.

Proposition 10.6. *For every $i \in \mathbb{Z}$ there are short exact sequences*

$$0 \to \mathrm{R}^1\lim_n H^{i-1}(X_{\acute{e}t}, \underline{\mathbb{Z}/\ell^n}) \to H^i(X_{pro\acute{e}t}, \underline{\mathbb{Z}_\ell}) \to \lim_n H^i(X_{\acute{e}t}, \underline{\mathbb{Z}/\ell^n}) \to 0.$$

Proof. Omitted. □

Remark 10.7. Compare the formula above with the formula 0.2 in [4].

Proposition 10.8. *If X is of finite type over a separably closed field, and if ℓ is invertible in X then*

$$H^i(X_{pro\acute{e}t}, \underline{\mathbb{Z}_\ell}) = \lim_n H^i(X_{\acute{e}t}, \underline{\mathbb{Z}/\ell^n})$$

for every i.

Proof. By Deligne's finiteness theorem ([2], chapitre 7, théorème 1.1) the cohomology groups $H^i(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n})$ are finite for all i. A projective system of finite groups automatically satisfies the Mittag-Leffler condition, so $\mathrm{R}^1\lim$ vanishes. □

11. A broader context

The computation from the previous section fits into a broader context presented in [1] subsection 3.4. Let us describe it.

Let $\mathcal{A} = \mathrm{Ab}(X_{\text{proét}})$, $x \in \mathbb{Z}$, and let $i_x \colon \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A}(\mathbb{N}))$ be the functor which sends a complex K to the diagram of complexes

$$\to K \xrightarrow{x} K \xrightarrow{x} K.$$

The functor i_x descends to the level of derived categories as an exact functor.

There is a natural transformation $\eta_x \colon i_x \to i_1$ defined by the diagram

$$\begin{array}{ccc} \downarrow & & \downarrow \\ K & \xrightarrow{x^3} & K \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow{x^2} & K \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} \\ K & \xrightarrow{x} & K. \end{array}$$

Let $p_x \colon \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A}(\mathbb{N}))$ be the functor which sends a complex K to the cone of $\eta_x \colon i_x(K) \to i_1(K)$. The functor p_x descends to the level of derived categories as an exact functor. By construction for each $K \in D(\mathcal{A})$ there is a canonical distinguished triangle $i_x(K) \to i_1(K) \to p_x(K) \to i_x(K)[1]$ natural in K.

Notice that $\pi_n(p_x(K)) = \mathrm{cone}(K \xrightarrow{x} K) = K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x^n}$, and that the natural morphisms $\pi_{n+1}(p_x(K)) \to \pi_n(p_x(K))$ come from restriction maps $\underline{\mathbb{Z}/x^{n+1}} \to \underline{\mathbb{Z}/x^n}$. Hence $\mathrm{Rlim}_n\, p_x(K)$ is a homotopy limit of the diagram

$$\to K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x^3} \to K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x^2} \to K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x}$$

where the transition maps are induced by restrictions. In the following we will write $\mathrm{Rlim}_n\, K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x^n}$ instead of $\mathrm{Rlim}\, p_x(K)$.

Since $\mathrm{Rlim}\, i_1(K) = K$ the natural map $i_1(K) \to p_x(K)$ induces a natural map $K \to \mathrm{Rlim}_n\, K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/x^n}$.

Definition 11.1. Let ℓ be a prime. An object $K \in D(X_{\text{proét}})$ is called derived ℓ-complete if the natural map $K \to \mathrm{Rlim}_n\, K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/\ell^n}$ is an isomorphism.

Writing $T(K, \ell) = \mathrm{Rlim}\, i_\ell(K)$ we obtain a canonical distinguished triangle

$$T(K,\ell) \to K \to \operatorname*{Rlim}_n K \otimes^L_{\underline{\mathbb{Z}}} \underline{\mathbb{Z}/\ell^n} \to T(K,\ell)[1].$$

So K is derived ℓ-complete if and only if $T(K, \ell) = 0$. From this it is clear that derived ℓ-complete objects form a full triangulated subcategory of $D(X_{\text{proét}})$.

If Y is another scheme, and $F\colon D(X_{\text{proét}}) \to D(Y_{\text{proét}})$ a functor which commutes with countable produts, then F sends dervided ℓ-complete objects to derived ℓ-complete objects. For example the derived pushforward $Rf_*\colon D(X_{\text{proét}}) \to D(Y_{\text{proét}})$ coming from a morphism $f\colon X \to Y$ commutes with products since it is right adjoint.

Similarly one introduces a notion of a derived ℓ-complete object in $D(\mathbb{Z})$ by replacing all the $\underline{\mathbb{Z}}$'s and $\underline{\mathbb{Z}/\ell^n}$'s above with $\mathbb{Z}$, and $\mathbb{Z}/\ell^n$ respectively. The functor $\mathrm{R}\Gamma\colon D(X_{\text{proét}}) \to D(\mathbb{Z})$ commutes with products, and so sends derived ℓ-complete objects to derived ℓ-complete ones.

In the previous section we first prove that $\underline{\mathbb{Z}_\ell}$ is derived ℓ-complete. As a consequence $\mathrm{R}\Gamma(X_{\text{proét}}, \underline{\mathbb{Z}_\ell})$ is derived ℓ-complete, and the exact sequence connecting $H^i(\mathrm{R}\Gamma(X_{\text{proét}}, \underline{\mathbb{Z}_\ell}))$ with the pro-étale cohomology of $\underline{\mathbb{Z}/\ell^n}$'s follows purely formally.

To avoid treating $D(X_{\text{proét}})$, and $D(\mathbb{Z})$ as different cases one can pass to the topoi $\mathrm{Sh}(X_{\text{proét}})$, and $\mathrm{Sh}(pt)$. Here the repleteness property enters the picture. Both $\mathrm{Sh}(X_{\text{proét}})$, and $\mathrm{Sh}(pt)$ are replete, so the theory of derived limits, and derived ℓ-complete objects in these topoi is a specialization of such a theory for replete topoi in general (section 3 of [1]).

References

[1] B. Bhatt, P. Scholze. *The pro-étale topology for schemes.* arXiv:1309.1198

[2] P. Deligne, *Séminaire de Géométrie Algébrique du Bois Marie - Cohomologie étale - (SGA $4\frac{1}{2}$).* LNM **569**. Springer, Berlin (1977)

[3] E. Freitag, R. Kiehl, *Etale Cohomology and the Weil Conjectures.* Springer-Verlag (1988)

[4] U. Jannsen, Continuous Étale Cohomology. *Math. Ann.* 280, 207 – 245. (1988)

[5] J. Jin, *The pro-étale site, part II* (this seminar)

[6] A.J. de Jong et al., *The Stacks Project*, http://stacks.math.columbia.edu

[7] A. Neeman. A counterexample to a 1961 "theorem" in homological algebra. *Invent. Math.* (2) 148, 397 – 420. (2002)

[8] A. Neeman. *Triangulated Categories.* Annals of Mathematics Studies **148**. Princeton University Press (2001)

[9] J.E. Roos. Derived Functors of Inverse Limits Revisited, *J. London Math. Soc.* (2) 73, 65 – 83. (2006)

[10] C. Weibel, *An Introduction to Homological Algebra.* Cambridge University Press (1997)