Elementary Number Theory - Exercise 10a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that, if $d = m^2$ is a square, then Pell's equation $x^2 - dy^2 = 1$ only has the trivial solutions $(x, y) = (\pm 1, 0)$.

Problem 2. Determine some non-trivial solutions of Pell's equation $x^2 - 2y^2 = 1$ and use them to find a rational approximation $\frac{x}{y}$ to $\sqrt{2}$ with $|\frac{x}{y} - \sqrt{2}| \leq 10^{-6}$.

Problem 3. Let $d > 0$ be a non-square. Consider the set

$$\mathbb{Q}(\sqrt{d}) = \{x + \sqrt{d}y \,:\, x, y \in \mathbb{Q}\}.$$

1. Show that $\mathbb{Q}(\sqrt{d})$ is closed under addition and multiplication.
2. Show that $(x + \sqrt{d}y)^{-1} = \frac{x - \sqrt{d}y}{x^2 - dy^2}$ for $(x, y) \neq (0, 0)$, and deduce that $\mathbb{Q}(\sqrt{d})$ is a field.
3. If $(x, y) \in \mathbb{Q}^2$ solves Pell's equation $x^2 - dy^2 = 1$, then we have

$$(x + \sqrt{d}y)^{-n} = (x - \sqrt{d}y)^n$$

   for every $n \in \mathbb{Z}$.

Problem 4. Compute the continued fraction expansion and the convergents of $\frac{128}{1527}$.

Problem 5. Show the identities

$$\begin{aligned}
[a_0, \dots, a_n] &= a_0 + \frac{1}{[a_1, \dots, a_n]}, \\
[a_0, \dots, a_n] &= [a_0, \dots, a_{n-1} + \tfrac{1}{a_n}], \\
[a_0, \dots, a_n] &= [a_0, \dots, a_n - 1, 1], \\
[a_0, \dots, a_n]^{-1} &= [0, a_0, \dots, a_n].
\end{aligned}$$

Problem 6 (sage)**.** Write programs that

1. finds the fundamental solution to Pell's equation (apply it to $d = 43$).
2. lists solutions to Pell's equation using Lagrange's Theorem.
3. computes the continued fraction expansion and the convergents of a rational number.