Elementary Number Theory - Exercise 11a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that, in a primitive Pythagorean triple (a, b, c), a, b, c are pairwise coprime, a and b have different parity, and c is odd.

Problem 2. Find all Pythagorean triples (a, b, c) with $c \leq 25$.

Problem 3. A Pythagorean triple (a, b, c) is called *almost isosceles* if $|a - b| = 1$.

1. Show that every almost isosceles Pythagorean triple is, up to switching a and b, of the form
$$\left(\frac{x-1}{2}, \frac{x+1}{2}, y\right)$$
where $(x, y) \in \mathbb{N}^2$ solves the *negative Pell equation* $x^2 - 2y^2 = -1$, and $x \geq 3$.
2. Show that every solution $(x, y) \in \mathbb{N}^2$ of $x^2 - 2y^2 = -1$ is of the form (x_n, y_n) where
$$x_n + \sqrt{2}y_n = (1 + \sqrt{2})^{2n+1}.$$
3. Determine the first three almost isosceles Pythagorean triples.

Problem 4. Fermat's Last Theorem states that for $n \geq 3$ the equation $a^n + b^n = c^n$ has no integer solution with a, b, c all different from 0. Show that it suffices to prove Fermat's Last Theorem for prime exponents $n = p \geq 3$.

Problem 5. Show that, for each $n \in \mathbb{N}$, the numbers
$$(2n + 1,\, 2n^2 + 2n,\, 2n^2 + 2n + 1)$$
form a primitive Pythagorean triple. Compute them for $n = 10^m$ for $m = 1, 2, 3, 4, 5$ and admire the beautiful pattern that you get.

Problem 6 (sage)**.** Write a program which lists all Pythagorean triples (a, b, c) with $c \leq N$ for a given N.