Elementary Number Theory - Exercise 12a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that the number of *ordered* partitions of n is given by 2^{n-1}.
Example: The ordered partitions of 3 are given by $3, 2+1, 1+2, 1+1+1$.

Problem 2. Determine the partition numbers $p(6), p_{\mathrm{d}}(6), p_{\mathrm{d}}^{\mathrm{even}}(6), p_{\mathrm{d}}^{\mathrm{odd}}(6)$, and $p(6,3)$, by listing the corresponding partitions.

Problem 3. Draw the Ferrers diagram of the partition $12 = 6+3+2+1$ and determine the conjugate partition.

Problem 4. Consider the following two sets of partitions of n:

$$S = \{(\lambda_1, \ldots, \lambda_k) : \lambda_1 = \lambda_2\}, \quad T = \{(\lambda_1, \ldots, \lambda_k) : \lambda_1, \ldots, \lambda_k \geq 2\}.$$

Show that $|S| = |T| = p(n) - p(n-1)$.

Problem 5. A partition is called *self-conjugate* if it is equal to its conjugate partition. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

Problem 6. Let $p(n,k)$ be the number of partitions of n with exactly k parts. Show the generating function identity

$$\sum_{n=0}^{\infty} p(n,k)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Hint: We have seen in the lecture that $p(n,k)$ also counts the number of partitions of n whose largest part equals k.

Problem 7 (sage)**.** Write programs which

1. list all partitions of n, and thereby compute $p(n)$,
2. compute $p(n)$ using the recusion $p(n,k) = p(n-1,k-1) + p(n-k,k)$.