Elementary Number Theory - Exercise 12b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Express $p(5)$ and $p(6)$ in terms of $p(0)$ and $p(1)$ using Euler's recursion. Then, compute $p(7)$ from $p(6), p(5), \dots, p(0)$.

Problem 2. Let $p_{\mathrm{d}}(n)$ be the number of partitions of n into distinct parts, and $p_{\mathrm{odd}}(n)$ the number of partitions of n into odd parts[1]. Prove *Euler's partition identity*

$$p_{\mathrm{d}}(n) = p_{\mathrm{odd}}(n),$$

by showing the generating function identities

$$\sum_{n=0}^{\infty} p_{\mathrm{d}}(n)x^n = \prod_{n=1}^{\infty}(1+x^n),$$
$$\sum_{n=0}^{\infty} p_{\mathrm{odd}}(n)x^n = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

Problem 3. Let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n. Show that its generating function is given by

$$\sum_{n=1}^{\infty} \sigma(n)x^n = \sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}.$$

Problem 4. The Fibonacci numbers F_m are defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_m = F_{m-1} + F_{m-2}.$$

For each $n \in \{1, \dots, 15\}$, count the number of partitions of n into distinct non-consecutive Fibonacci numbers. Make a conjecture based on your results[2].

Problem 5 (sage)**.** Write a program that computes $p(n)$ using Euler's recursion. Use it to compute $p(100)$.

[1] As usual, we put $p_{\mathrm{d}}(0) = p_{\mathrm{odd}}(0) = 1$.
[2] Look up Zeckendorf's Theorem to validate your conjecture.