# Elementary Number Theory - Exercise 1b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

**Problem 1.** Let $a, b, n \in \mathbb{N}$ be natural numbers.

1. What is $\gcd(n, 0), \gcd(n, 1), \gcd(n, n), \gcd(n, 2n)$?
2. Show that $\gcd(a, a + b) = \gcd(a, b)$.
3. Show that $\gcd(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}) = 1$.

**Problem 2.**

1. Show that $\gcd(n, n + 1) = 1$ for any $n \in \mathbb{Z}$.
2. Show that $\gcd(22n + 7, 33n + 10) = 1$ for any $n \in \mathbb{Z}$.

**Problem 3.** Show that, for $a = bq + r$, we have $\gcd(a, b) = \gcd(b, r)$. Use this to convince yourself that the Euclidean Algorithm really computes $\gcd(a, b)$.

**Problem 4.** Compute $\gcd(90, 14)$ and $x, y \in \mathbb{Z}$ with $90x + 14y = \gcd(90, 14)$ using the Euclidean Algorithm.

**Problem 5.** (Homework) The Fibonacci numbers $F_n$ are defined recursively via $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The first few $F_n$ are given by $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

1. Show that
$$\gcd(F_n, F_{n+1}) = 1, \qquad \text{for all } n \in \mathbb{N}.$$
2. Prove *Honsberger's identity*
$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n, \qquad \text{for all } m, n \in \mathbb{N}.$$
3. Show that $m \mid n$ implies that $F_m \mid F_n$ .

**Problem 6.** (Homework)

1. Find three integers which are coprime, but not pairwise coprime.
2. For any given $r$, find a sequence $a_1, a_2, \ldots, a_r$ of $r$ integers which are coprime (that is, $\gcd(a_1, \ldots, a_r) = 1$), such that any $r - 1$ elements of the sequence are not coprime.
   *Remark:* The numbers $a_1, \ldots, a_r$ need not be *pairwise* coprime.

**Problem 7** (sage)**.** Implement the following functions in sage:

1. `divide(a,b)`: compute $q, r$ such that $a = bq + r$ and $0 \leq r < q$.
2. `gcd(a,b)`: compute $\gcd(a, b)$ using the Euclidean Algorithm.
3. `bezout(a,b)`: compute integers $x, y$ such that $ax + by = \gcd(a, b)$.