Elementary Number Theory - Exercise 2a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1.

1. Show that
$$\prod_{m+1<p\leq 2m+1} p \leq \binom{2m+1}{m}.$$
Hint: Which primes appear in the numerator and denominator of $\binom{2m+1}{m}$?

2. Show that
$$\binom{2m+1}{m} \leq 2^{2m}.$$
Hint: Rewrite 2^{2m+1} using the Binomial Theorem.

Problem 2. Show that the multiplicity of p in $n!$ is
$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Problem 3. Let $n \geq 3$. Show that, for $\frac{2}{3}n < p \leq n$, we have $p \nmid \binom{2n}{n}$.
Hint: How often does p appear in the numerator and denominator of $\binom{2n}{n}$?

Problem 4. Show that
$$\binom{2n}{n} \geq \frac{2^{2n}}{2n}.$$
Hint: Binomial Theorem.

Problem 5 (sage)**.** Convince yourself numerically that we have
$$4^{\frac{1}{3}n} > (2n)^{1+\sqrt{2n}}$$
for $n > 4000$. For example, you could plot both functions. You could also try wolframalpha.com for this task! What would be a better threshold than 4000?