Elementary Number Theory - Exercise 2b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Determine $\pi(100)$.

Problem 2. Show that $(3, 5, 7)$ is the only triple of primes of the form $(n, n+2, n+4)$.

Problem 3. Show that, if $p > 3$ and $q = p + 2$ are twin primes, then $p + q$ is divisible by 12.

Problem 4. Show that there are infinitely many primes whose leading digit is 1.
Hint: How do intervals containing only numbers with leading digit 1 look like?

Problem 5. Let $\mathbb{P} = \{2, 3, 5, 7, 11, \dots\}$ be the set of primes. Show that every natural number $n > 6$ can be written as a sum of distinct elements from $\mathbb{P} \cup \{1\}$.
Hint: Use Bertrand's Postulate.

Problem 6 (Homework)**.**

1. Using the Prime Number Theorem, show the following strengthening of Bertrand's Postulate: For any $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a prime p with $n < p \leq (1+\varepsilon)n$.
2. Use the above to show that, for any given number m, there exist infinitely many primes starting with the digits of m. For example, there exist infinitely many primes starting with 313, e.g. $313, 3137, 31357, 313471, \dots$.

Problem 7 (Homework)**.** Let p_n be the n-th prime. The n-th *primorial* $p_n\#$ is defined as the product of the primes up to p_n,

$$p_1\# = 2, \quad p_2\# = 2 \cdot 3 = 6, \quad p_3\# = 2 \cdot 3 \cdot 5 = 30, \quad p_4\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210, \quad \dots$$

The n-th *Fortunate number*[1] F_n is the gap between $p_n\#$ and the next prime after $p_n\# + 1$. For example, the next prime after $p_2\# + 1 = 7$ is 11, so $F_2 = 11 - p_2\# = 11 - 6 = 5$.

Compute the first 10 Fortunate numbers[2]. Do you notice something special about them[3]?

Problem 8 (sage)**.**

1. Implement the prime counting function $\pi(x)$.
2. Let $\psi(x)$ be the number of primes of the form $4k+3$ that are $\leq x$. Implement $\psi(x)$ and make a guess what $\lim_{x\to\infty} \psi(x)/\pi(x)$ could be.
3. Write a program that prints a list of Fortunate numbers (bonus: get rid of duplicates and list the Fortunate numbers in ascending order), and check your conjecture from the last problem.

[1] Named after the social anthropologist Reo Fortune.
[2] You might want to use sageMath or Wolframalpha for this
[3] You may look up *Fortune's Conjecture* to validate your findings