Elementary Number Theory - Exercise 3a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Let f be a multiplicative number-theoretic function which is not the constant 0-function. Show that $f(1) = 1$.

Problem 2.

1. Let $m, n \in \mathbb{N}$ be coprime. Show that every divisor $d \mid mn$ can be written uniquely as $d = d_1 d_2$ with $d_1 \mid m$ and $d_2 \mid n$.
2. Let $f \neq 0$ be a multiplicative number-theoretic function. Show that the summatory function of f,
$$F(n) = \sum_{d \mid n} f(d)$$
is multiplicative, as well.

Problem 3. We consider the divisor sum $\sigma_k(n) = \sum_{d \mid n} d^k$ for $k \in \mathbb{N}$.

1. Show that $\sigma_k(n)$ is multiplicative, but not completely multiplicative.
2. Let $n = p^m$ be a power of a prime p. Show that
$$\sigma_k(p^m) = \frac{p^{k(m+1)} - 1}{p^k - 1}.$$
3. Write down an explicit formula for $\sigma_k(n)$ in terms of the prime factorization of n.
4. Compute $\sigma_1(24)$ using the definition, and using your formula from the last item.

Problem 4. Let

$$\mu(n) = \begin{cases} (-1)^r, & \text{if } n \text{ is square-free, } n = p_1 \cdots p_r, \\ 0, & \text{otherwise,} \end{cases}$$

be the Möbius function. Check that μ is multiplicative, but not completely multiplicative, and show that

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Use that $F(n) = \sum_{d \mid n} \mu(d)$ is multiplicative, and reduce the problem to the prime powers dividing n.

Problem 5. (Homework) The Liouville function is defined by $\lambda(1) = 1$ and

$$\lambda(n) = (-1)^{\nu_1 + \ldots + \nu_r}, \quad \text{for } n = p_1^{\nu_1} \cdots p_r^{\nu_r}.$$

Check that λ is completely multiplicative and show that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise} \end{cases}$$

Hint: First show the identity for prime powers n, and then use multiplicativity.

Problem 6 (sage)**.** Implement the following functions in sage:

1. `sigma(k,n)`: the divisor sum $\sigma_k(n)$.
2. `moebius(n)`: the Moebius function $\mu(n)$.
3. `phi(n)`: Euler's totient function $\varphi(n)$.