Elementary Number Theory - Exercise 3b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Let $\mathbf{1}(n) = 1$ be the constant 1-function, $\mathrm{id}(n) = n$ the identity, and $e(n) = 1$ for $n = 1$ and $e(n) = 0$ for $n > 1$. Compute the following Dirichlet convolutions:

$$\mathbf{1} * \mathbf{1}, \qquad \mathbf{1} * \mathrm{id}, \qquad \mathbf{1} * e, \qquad \mathrm{id} * \mathrm{id}.$$

Problem 2. Let $\tau(n) = \sum_{d|n} 1$ be the number-of-divisors function and $\sigma(n) = \sum_{d|n} d$ the sum-of-divisors function, and let φ Euler's totient function. Show that

$$\sum_{d|n} \tau\left(\frac{n}{d}\right)\varphi(d) = \sigma(n).$$

Hint: Use that $\varphi * \mathbf{1} = \mathrm{id}$ and convolute with $\mathbf{1}$.

Problem 3. Let

$$\varphi(n) = \#\{1 \leq k \leq n \mid \gcd(k, n) = 1\}$$

be Euler's totient function.

1. Show that, if $n = p^m$ is a power of a prime, then

$$\varphi(p^m) = p^{m-1}(p-1).$$

   Hint: Count the elements $1 \leq k \leq p^m$ which are *not* coprime to p.

2. We know from the lecture that φ is multiplicative. Conlude that $\varphi(n)$ is given by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

   where the product runs over the prime divisors of n.

Problem 4. Show that, for a *completely* multiplicative number-theoretic function $f \neq 0$, the inverse $\hat{f}$ with respect to convolution is given by

$$\hat{f}(n) = \mu(n) f(n), \qquad n \in \mathbb{N}.$$

Problem 5.

1. Show that

$$\varphi(n) = (\mu * \mathrm{id})(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

   Hint: Apply the Möbius inversion formula to the summatory function of φ.

2. Show that the inverse $\hat{\varphi}$ of φ with respect to convolution is given by

$$\hat{\varphi}(n) = \sum_{d|n} \mu(d)d = \prod_{p|n}(1-p).$$

Hint: We have $\hat{\varphi} = \hat{\mu} * \hat{\mathrm{id}}$ by the first item. What are $\hat{\mu}$ and $\hat{\mathrm{id}}$?

Problem 6 (Homework)**.** Show that the Dirichlet-inverse $\hat{\sigma}_k(n)$ of the divisor sum $\sigma_k(n)$ is the multiplicative function which is given on prime powers p^m by

$$\hat{\sigma}_k(p^m) = \begin{cases} -p^k - 1, & \text{if } m = 1, \\ p^k, & \text{if } m = 2, \\ 0, & \text{if } m \geq 3. \end{cases}$$

Problem 7 (sage)**.** Write a function that

1. computes the Dirichlet convolution of two number-theoretic functions f, g.
2. computes the Dirichlet-inverse $\hat{f}$ of a number-theoretic function f.