Elementary Number Theory - Exercise 4a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that if m is composite, then $2^m - 1$ is composite.
Hint: If $m = ab$, show that $2^a - 1 \mid 2^m - 1$.

Problem 2. Show that a natural number of the form $n = 2^{m-1}(2^m - 1)$, with $2^m - 1$ being prime, is perfect.

Problem 3. Check that 6 and 28 are perfect, and find the next two even perfect numbers.

Problem 4. Show that every even perfect number ends with the digit 6 or 8.

Problem 5. Show that every even perfect number n can be written as the sum of the first d natural numbers, $n = 1 + 2 + 3 + \ldots + d$, for a suitable $d \in \mathbb{N}$.

Problem 6. We have seen that an *odd* perfect number must be of the form $n = p^{2m+1}Q^2$ with an odd prime p and an odd natural number Q with $\gcd(p, Q) = 1$.

Show that p must be of the form $p = 4k + 1$, and m must be even.
Hint: Use that $\sigma(p^{2m+1}) = (1 + p)(1 + p^2 + p^4 + \cdots + p^{2m})$.

Problem 7. (Homework) Prove Thabit's rule and apply it to $k \in \{1, \ldots, 10\}$ to find three pairs of amicable numbers[1].

Problem 8 (Homework)**.** The *harmonic mean* of the divisors of a natural number n is defined by

$$H(n) = \frac{n\tau(n)}{\sigma(n)}.$$

For example, $H(4) = \frac{4\tau(4)}{\sigma(4)} = \frac{4\cdot 3}{7} = \frac{12}{7}$. A number n is called *harmonic* if the harmonic mean $H(n)$ of its divisors is an integer.

1. Check that 6 and 140 are harmonic numbers.
2. Show that every perfect number is harmonic.

Problem 9 (sage)**.** Write programs that find

1. Mersenne primes,
2. perfect numbers,
3. pairs of amicable numbers.
   Bonus: Find some pairs that are not covered by Thabit's rule.

[1] Until today, only these three pairs of amicable numbers satisfying Tabit's rule have been found.