Elementary Number Theory - Exercise 4b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that

$$\sum_{d|n} \sigma\left(\frac{n}{d}\right)\varphi(d) = n\tau(n).$$

Problem 2. Recall Liouville's λ-function defined by $\lambda(1) = 1$ and

$$\lambda(n) = (-1)^{\nu_1+\ldots+\nu_r}, \quad \text{for } n = p_1^{\nu_1}\cdots p_r^{\nu_r}.$$

Show that the Dirichlet-inverse of λ is given by $\hat{\lambda}(n) = |\mu(n)|$.

Problem 3. Show that $\tau^3 * \mathbf{1} = (\tau * \mathbf{1})^2$.

Problem 4. Show that

$$\tau(mn) = \sum_{d|\gcd(m,n)} \mu(d)\tau(m/d)\tau(n/d)$$

for all $m, n \in \mathbb{N}$.

Problem 5. A number n is called *abundant* if $\sigma(n) > 2n$, and *deficient* if $\sigma(n) < 2n$ (recall that n is called *perfect* if $\sigma(n) = 2n$).

1. Show that every multiple of a perfect number (except the perfect number itself) is abundant[1], and every proper divisor of a perfect number is deficient.
2. Show that every prime power is deficient[2].
3. Numbers n with $\sigma(n) = 2n + 1$ (or $2n - 1$) are called *slightly abundant* (or *slightly deficient*). Find infinitely many examples of slightly deficient numbers[3].

Problem 6. Show that every even perfect number $n > 6$ can be written as a sum of the first d odd cubes, $n = 1^3 + 3^3 + 5^3 + \ldots + d^3$, for a suitable odd $d \in \mathbb{N}$. For example, $28 = 1^3 + 3^3$. *Hint:* Use the summation formula $1^3 + 2^3 + 3^3 + 4^3 + \ldots + (d-1)^3 + d^3 = \frac{d^2(d+1)^2}{4}$.

[1] In particular, there are infinitely many abundant numbers.

[2] In particular, there are infinitely many deficient numbers.

[3] It is not known whether any slightly abundant numbers exist!