Elementary Number Theory - Exercise 5b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. If p is an odd prime, show that $x^2 \equiv 1 \pmod{p}$ has exactly 2 incongruent solutions modulo p.

Problem 2. Modulo 101, how many roots are there to the polynomial equation

$$x^{99} + x^{98} + \cdots + x + 1 \equiv 0 \pmod{101}?$$

Hint: Multiply with $x(x-1)$.

Problem 3. Show that, if $n > 4$ is composite, then n divides $(n-1)!$.

Problem 4. Let p be a prime. Wilson's Theorem tells us that $(p-1)! + 1 = kp$ for some $k \in \mathbb{N}$. When is $k = 1$ or $k = p$?

Problem 5. Find the remainder when 98! is divided by 101.

Problem 6. Compute $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$ for $p = 7$ and $p = 11$ to convince yourself that Wolstenholme's Theorem works.

Problem 7. Let $p > 3$ be a prime. Show that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

Hint: Relate the binomal coefficient to a special value of $h(x) = (x-1)(x-2)\cdots(x-(p-1))$ and use the first claim in Wolstenholme's Theorem.

Problem 8 (sage)**.** Check the following conjecture numerically: If $p > 3$ is prime, and we write $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1} = \frac{a}{b}$ with $\gcd(a, b) = 1$, then $\frac{a}{p^2}$ is square-free.