Elementary Number Theory - Exercise 8a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. For each n $= 1, 2, \dots, 15$, check if n is a sum of two or three squares.

Problem 2. Write 45 and 585 as sums of two squares.
Hint: Diophantus' two squares identity.

Problem 3. Show that, if n can be written as a sum of three squares, then n cannot be of the form $4^a(8b+7)$ with non-negative integers a, b.

Problem 4. We let

$$r_4(n) = \#\{(a,b,c,d) \in \mathbb{Z}^2 \, : \, n = a^2 + b^2 + c^2 + d^2\}$$

be the number of ways to write n as sum of four squares. Show that $r_4(n)$ is divisible by 8.

Problem 5. Show that every natural number can be written as a sum of five integer *cubes*. To this end, show that $n^3 \equiv n \pmod 6$, hence $n^3 - n = 6k$, and check that

$$n = n^3 + k^3 + k^3 + (-k-1)^3 + (1-k)^3.$$

Write $n = 7$ as a sum of five cubes. However, convince yourself that 7 cannot be written as a sum of five cubes of *non-negative* integers.

Problem 6. Show that, if an odd prime p can be written in the form $p = x^2 + 2y^2$, then -2 is a square modulo p.

Problem 7. Which integers n can be written in the form $n = x^2 - y^2$?

Problem 8 (sage)**.**

1. Write a program to find representations of an odd prime p as $p = x^2 + 2y^2$, and use it to numerically verify that p can be written in this way if and only if -2 is a square modulo p.
2. Write a program that counts $r_4(n)$. Use it to numerically verify Jacobi's formula

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d.$$