Elementary Number Theory - Exercise 8b

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Problem 1. Consider the quadratic forms

$$Q_1 = [1, 2, 3], \qquad Q_2 = [2, 4, 3], \qquad Q_3 = [1, 3, 1].$$

1. Write down the Gram matrices of Q_1, Q_2 and Q_3.
2. Compute the discriminants of Q_1, Q_2, and Q_3.
3. Which of these forms is positive definite or indefinite?
4. Show that Q_1 is equivalent to Q_2 via

$$Q_2 = Q_1 \circ \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$$

5. For each of the three forms, find three different integers that they represent.

Problem 2. Show that $Q = [a, b, c]$ properly represents a, c, and $a + b + c$.

Problem 3. Let $Q = [a, b, c]$ be a quadratic form.

1. Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Show that $T^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and compute

$$Q \circ T^n = [a, b + 2an, an^2 + bn + c], \qquad Q \circ S = [c, -b, a].$$

2. Let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and write

$$Q \circ M = M^t Q M = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}.$$

   Show that a', b', c' are explicitly given by

$$\begin{aligned} a' &= a\alpha^2 + b\alpha\gamma + c\gamma^2 \\ b' &= 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta \\ c' &= a\beta^2 + b\beta\delta + c\delta^2. \end{aligned}$$

Problem 4. Let $Q = [a, b, c]$ and $Q' = [a', b', c']$ be equivalent. Show that

1. Q and Q' (properly) represent the same integers.
2. Q and Q' have the same discriminant.

3. Q is positive definite (resp. indefinite) if and only if Q' is positive definite (resp. indefinite).

4. Q is primitive if and only if Q' is primitive.

Problem 5. Show that a quadratic form properly represents an integer n if and only if it is equivalent to a form of the shape $[n, b', c']$ for some $b', c' \in \mathbb{Z}$.
Hint: Use the explicit formula for the coefficients of $Q \circ M$ derived above.

Problem 6 (sage)**.** A quadratic form can be represented in sage as an array $Q = [a, b, c]$. Write programs that

1. compute the discriminant of Q,

2. check whether Q is positive (resp. negative) definite or indefinite,

3. check whether Q is primitive,

4. compute $Q \circ M$ for $M \in \mathrm{SL}_2(\mathbb{Z})$.