# Elementary Number Theory - Exercise 10b

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**Problem 1.** Show that $\pi = [3, 7, 15, 1, \dots]$ and use it to find the rational approximation

$$\frac{355}{113} = 3.14159292035398$$

of $\pi = 3.14159265358979....$

**Solution 1.** We compute

$$\begin{aligned}
\pi = 3.14159265 = 3 + 0.14159265 &= 3 + \frac{1}{7.06251348} \\
&= 3 + \frac{1}{7 + 0.06251348} = 3 + \frac{1}{7 + \frac{1}{15.9965498641253}} \\
&= 3 + \frac{1}{7 + \frac{1}{15 + 0.9965498641253}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + 0.00346208052291}}} \\
&= 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + 0.00346208052291}}} = [3, 7, 15, 1, \dots].
\end{aligned}$$

Now the third convergent of $\pi$ is given by

$$c_3 = [3, 7, 15, 1] = \frac{355}{113}.$$

Since the convergents converge to $\pi$, the $c_n$ yield good approximations to $\pi$ for large $n$. Since 0.00346208052291 is quite small, the third convergent $c_3$ yields an exceptionally good approximation.

**Problem 2.** Compute the continued fraction expansions of $\sqrt{5}$ and the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.

**Solution 2.** We have $\sqrt{5} = 2.236...$ so

$$\sqrt{5} = 2 + (\sqrt{5} - 2) = 2 + \frac{1}{\frac{1}{\sqrt{5}-2}} = 2 + \frac{1}{\sqrt{5} + 2}.$$

Now $\sqrt{5} + 2 = 4.236...$ so

$$\sqrt{5} = 2 + \frac{1}{\sqrt{5} + 2} = 2 + \frac{1}{4 + (\sqrt{5} - 2)} = 2 + \frac{1}{4 + \frac{1}{\frac{1}{\sqrt{5}-2}}} = 2 + \frac{1}{4 + \frac{1}{\sqrt{5}+2}}.$$

We see that the algorithm continues from now on, so we find

$$\sqrt{5} = [2, 4, 4, 4, 4, \dots] = [2, \overline{4}].$$

For the golden ratio, one can just go through the algorithm to compute the continued fraction expansion, to see that the algorithm repeats itself already after the first step, and yields

$$\phi = [1, 1, 1, \dots] = [\,\overline{1}\,].$$

Alternatively, one can use that $\phi = 1 + \frac{1}{\phi}$:

$$\frac{1}{\phi} = \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{2(1-\sqrt{5})}{-4} = \frac{-1+\sqrt{5}}{2} = \phi - 1,$$

which also gives

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1+\frac{1}{\phi}} = 1 + \frac{1}{1+\frac{1}{1+\phi}} = [1, 1, 1, 1, \dots] = [\,\overline{1}\,].$$

**Problem 3.** Which quadratic irrational has the continued fraction expansion $[1, \overline{6, 2}]$?

**Solution 3.** We write

$$x = [1, \overline{6, 2}] = 1 + \frac{1}{[\,\overline{6, 2}\,]}.$$

We determine the number $\alpha = [\,\overline{6, 2}\,]$. We have,

$$\alpha = 6 + \frac{1}{2 + \frac{1}{\alpha}},$$

which can be simplified to

$$2\alpha^2 - 12\alpha - 6 = 0.$$

This has the roots

$$\alpha = \frac{6 \pm \sqrt{48}}{2} = 3 \pm 2\sqrt{3}.$$

Since $\alpha > 0$, we find

$$\alpha = 3 + 2\sqrt{3},$$

and then

$$x = 1 + \frac{1}{3+2\sqrt{3}} = 1 + \frac{3-2\sqrt{3}}{9-12} = 1 + \frac{2\sqrt{3}-3}{3} = \frac{2\sqrt{3}}{3}.$$

Applying the continued fraction algorithm to $x$, one can check that we indeed have $x = [1, \overline{6, 2}]$.

**Problem 4.** Compute the continued fraction expansion of $\sqrt{10}$ and use it to determine the fundamental solution of Pell's equation $x^2 - 10y^2 = 1$.

**Solution 4.** We have $\sqrt{10} = [3, \overline{6}]$, so the period is $n = 1$. Since $n$ is odd, we consider the convergent $c_{2n-1} = c_1$, given by

$$c_1 = [3, 6] = 3 + \frac{1}{6} = \frac{19}{6}.$$

We find that $(19, 6)$ is the fundamental solution to Pell's equation $x^2 - 10y^2 = 1$.

**Problem 5** (sage)**.** Write a program that computes the continued fraction expansion of $\sqrt{d}$ and gives a fundamental solution to Pell's equation. Verify that

$$\sqrt{61} = [7, \overline{1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14}]$$

and solve Pell's equation $x^2 - 61y^2 = 1$.