Elementary Number Theory - Exercise 11b

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that 2023 is a congruent number.

Solution 1. We have $2023 = 7 \cdot 17^2$, and 7 is a congruent number using the triangle $(24/5, 35/12, 337/60)$, so 2023 is also a congruent number.

Problem 2. In this problem, we show that $13, 14, 15$ are congruent numbers, using different approaches.

1. Show that 13 is a congruent number, using the triangle with side lengths
$$(104329, 23400, 106921).$$
2. Show that 14 is a congruent number, using that $(x, y) = (18, 48)$ is a rational point on the elliptic curve $y^2 = x^3 - 14^2x$.
3. Show that 15 is a congruent number by finding a Pythagorean triple (a, b, c) with area $ab/2 = 60$.

Solution 2.

1. We have
$$104329^2 + 23400^2 = 106921^2$$
and
$$104329 \cdot 23400 = 1220649300 = 13 \cdot 9690^2,$$
so 13 is a congruent number.
2. We indeed have
$$48^2 = 18^3 - 14^2 \cdot 18,$$
so the given point really lies on the elliptic curve. We have seen in the lecture that we have a bijection between rational point (x, y) on $y^2 = x^3 - n^2x$ with $y > 0$ and the rational right triangles with area n. The rational right triangle corresponding to $(112, 1176)$ is given by
$$(a, b, c) = \left(\frac{18^2 - 14^2}{48}, \frac{2 \cdot 14 \cdot 18}{48}, \frac{18^2 + 14^2}{48}\right) = \left(\frac{8}{3}, \frac{21}{2}, \frac{65}{6}\right).$$
Indeed, we have $\left(\frac{21}{2}\right)^2 + \left(\frac{8}{3}\right)^2 = \left(\frac{65}{6}\right)^2$ and area $\frac{1}{2} \cdot \frac{21}{2} \cdot \frac{8}{3} = 14$.
3. In the last exercise we found all Pythagorean triples with $c \leq 25$, and one of them was $(15, 8, 17)$. The corresponding right triangle has area $15 \cdot 8/2 = 60 = 4 \cdot 15$. Hence 60 is a congruent number. Since 4 is a square, $15 = 60/4$ is also a congruent number.

Problem 3. Use Tunnell's Theorem to determine the congruent numbers ≤ 15.

Solution 3. Fermat has shown that $1, 2, 3$ and all their square multiples are not congruent. Hence $1, 2, 3, 4, 8, 9, 12$ are not congruent. We have seen rational right triangles for $5, 6, 7$ in the lecture, and we have seen above that $13, 14, 15$ are congruent. It remains to check 10 and 11.

We have

$$C(10) = D(10) = 4,$$
$$A(11) = 12,\ B(11) = 4.$$

Hence, Tunnell's Theorem tells us that 10 and 11 are not congruent. In total, the congruent numbers ≤ 15 are given by

$$5, 6, 7, 13, 14, 15.$$

Problem 4. Show that, if the converse of Tunnell's Theorem can be proved to be true (e.g. if the weak BSD conjecture is true), then every natural number $n \equiv 5, 6, 7 \pmod 8$ is a congruent number.

Solution 4. If $n \equiv 5, 7 \pmod 8$, then n is odd, so we need to check that $A(n) = 2B(n)$ in Tunnell's Theorem. In fact, we have $A(n) = 0$ and $B(n) = 0$. Indeed, any solution of the quadratic equations $x^2 + 2y^2 + 8z^2 = n$ or $x^2 + 2y^2 + 32z^2 = n$ would give a solution to $x^2 + 2y^2 \equiv 5, 7 \pmod 8$, which is not possible. Hence $A(n) = 0 = 2B(n)$, so if the converse of Tunnell's Theorem is true, then it tells us that n is a congruent number.

Similarly, if $n \equiv 6 \pmod 8$, then n is even, so we need to check $C(n) = 2D(n)$. Again, we even have $C(n) = 0$ and $D(n) = 0$. Any solution of $8x^2+2y^2+16z^2 = n$ or $8x^2+2y^2+64z^2 = n$ would give a solution to $2x^2 \equiv 6 \pmod 8$, hence $x^2 \equiv 3 \pmod 4$, which is impossible.

Problem 5 (sage)**.** Using Tunnell's Theorem, determine the congruent numbers ≤ 100. For each of the numbers which might be congruent, find a suitable rational right triangle to verify that they are indeed congruent.