Elementary Number Theory - Exercise 12a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Show that the number of *ordered* partitions of n is given by 2^{n-1}.
Example: The ordered partitions of 3 are given by $3, 2+1, 1+2, 1+1+1$.

Solution 1. Every ordered partition corresponds to a way to put some slashes | in a sequence of 1's of length n, e.g. the partitions $6 = 3+1+2$ and $6 = 2+1+2+1$ corresponds to

$$1\,1\,1\,|\,1\,|\,1\,1 \quad \text{and} \quad 1\,1\,|\,1\,|\,1\,1\,|\,1$$

Note that we only need to put slashes in between two 1's but not in front of the first 1 or after the last 1, so there are $n-1$ places where we can put a dash. At each of these $n-1$ places, the are two possibilities: a dash or no dash. This gives 2^{n-1} possible ways to put dashes between the 1's, hence there are 2^{n-1} ordered partitions.

Problem 2. Determine the partition numbers $p(6), p_{\mathrm{d}}(6), p_{\mathrm{d}}^{\mathrm{even}}(6), p_{\mathrm{d}}^{\mathrm{odd}}(6)$, and $p(6,3)$, by listing the corresponding partitions.

Solution 2. We list the partitions of 6

$$\begin{aligned} 6 &= 5+1 \\ &= 4+2 \\ &= 4+1+1 \\ &= 3+3 \\ &= 3+2+1 \\ &= 3+1+1+1 \\ &= 2+2+2 \\ &= 2+2+1+1 \\ &= 2+1+1+1+1 \\ &= 1+1+1+1+1+1 \end{aligned}$$

Hence we have

$$p(6) = 11, \quad p_{\mathrm{d}}(6) = 4, \quad p_{\mathrm{ed}}(6) = 2, \quad p_{\mathrm{od}}(6) = 2, \quad p(6,3) = 3.$$

Problem 3. Draw the Ferrers diagram of the partition $12 = 6+3+2+1$ and determine the conjugate partition.

Solution 3. Omitted!

Problem 4. Consider the following two sets of partitions of n:

$$S = \{(\lambda_1, \dots, \lambda_k) \,:\, \lambda_1 = \lambda_2\}, \quad T = \{(\lambda_1, \dots, \lambda_k) \,:\, \lambda_1, \dots, \lambda_k \geq 2\}.$$

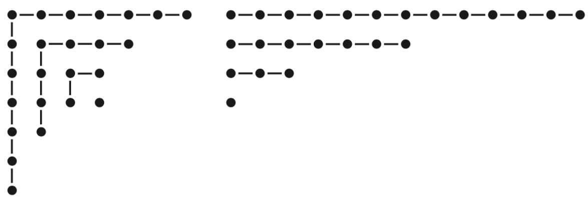
Show that $|S| = |T| = p(n) - p(n-1)$.

Solution 4. We claim that conjugation yields a bijection from S to T. Indeed, in the Ferrers diagram of a partition in S, the first two rows have the same number of dots, so in the conjugate partition, the first two columns have the same number of dots, which means that each row has at least two dots.

Note that the set T consists precisely of those partitions which have no part equal to 1. Hence, $|T|$ is equal to the number of all partitions of n, minus the number of partitions with at least one part equal to 1. If we omit the last one in such a partition $(\lambda_1, \dots, \lambda_{k-1}, 1)$ of n, we obtain a partition $(\lambda_1, \dots, \lambda_{k-1})$ of $n-1$. Conversely, by adding a 1 at the end, each partition of $n-1$ yields a partition of n with at least one part equal to 1. Hence, we have a bijection between the partitions of n with at least one part equal to 1, and the set of all partitions of $n-1$. This shows $|T| = p(n) - p(n-1)$.

Problem 5. A partition is called *self-conjugate* if it is equal to its conjugate partition. Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

Solution 5. In the Ferrers diagram of a self-conjugate partition, the "shells" outlined as in the left picture below always consist of an odd number of dots. Hence, a self-conjugate partition yields a partition into odd parts. This process is indicated by the picture for the self-conjugate partition $24 = 7+5+4+4+2+1+1$ and the odd-parts partition $24 = 13+7+3+1$



Conversely, every partition into odd parts yields a self-conjugate partition, by using the rows of the odd-parts partition as the shells of the self-conjugate partition.

Problem 6. Let $p(n,k)$ be the number of partitions of n with exactly k parts. Show the generating function identity

$$\sum_{n=0}^{\infty} p(n,k)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Hint: We have seen in the lecture that $p(n,k)$ also counts the number of partitions of n whose largest part equals k.

Solution 6. Using the geometric series we can write the right-hand side as

$$\begin{aligned}\frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)} &= x^k\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty} x^{2n}\right)\cdots\left(\sum_{n=0}^{\infty} x^{kn}\right)\\ &= x^k\left(1+x^{1\cdot 1}+x^{2\cdot 1}+x^{3\cdot 1}+x^{4\cdot 1}\dots\right)\\ &\quad\cdot\left(1+x^{1\cdot 2}+x^{2\cdot 2}+x^{3\cdot 2}+x^{4\cdot 2}\dots\right)\\ &\quad\vdots\\ &\quad\cdot\left(1+x^{1\cdot k}+x^{2\cdot k}+x^{3\cdot k}+x^{4\cdot k}\dots\right).\end{aligned}$$

Hence the coefficient at x^n is gets a contribution $+1$ from each products of monomials of the form

$$x^k\cdot x^{n_1\cdot 1}\cdot x^{n_1\cdot 2}\cdots x^{n_k\cdot k}$$

with

$$k+n_1\cdot 1+n_1\cdot 2+\cdots+n_k\cdot k=n.$$

The above tuples $(n_1,\dots,n_k)$ correspond to the partitions

$$n=k+\underbrace{(1+\cdots+1)}_{n_1\text{ times}}+\underbrace{(2+\cdots+2)}_{n_2\text{ times}}+\cdots+\underbrace{(k+\cdots+k)}_{n_k\text{ times}},$$

which are precisely the partitions of n whose largest part equals k. Hence, the coefficient at x^n is given by $p(n,k)$.

Problem 7 (sage)**.** Write programs which

1. list all partitions of n, and thereby compute $p(n)$,
2. compute $p(n)$ using the recusion $p(n,k)=p(n-1,k-1)+p(n-k,k)$.