Elementary Number Theory - Exercise 12b

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Problem 1. Express $p(5)$ and $p(6)$ in terms of $p(0)$ and $p(1)$ using Euler's recursion. Then, compute $p(7)$ from $p(6), p(5), \ldots, p(0)$.

Solution 1. We have

$$\begin{aligned}
p(6) &= p(5) + p(4) - p(1) \\
&= (p(4) + p(3) - p(0)) + (p(3) + p(2)) - p(1) \\
&= ([p(3) + p(2)] + [p(2) + p(1)] - p(0)) + ([p(2) + p(1)] + [p(1) + p(0)]) - p(1) \\
&= ([[p(2) + p(1)] + [p(1) + p(0)]] + [[p(1) + p(0)] + p(1)] - p(0)) \\
&\quad + ([[p(1) + p(0)] + p(1)] + [p(1) + p(0)]) - p(1) \\
&= 7p(1) + 4p(0).
\end{aligned}$$

From this, we also get

$$p(5) = ([[p(2) + p(1)] + [p(1) + p(0)]] + [[p(1) + p(0)] + p(1)] - p(0)) = 5p(1) + 2p(0).$$

Using $p(1) = p(0) = 1$, we get $p(6) = 11$ and $p(5) = 7$.

We have

$$p(7) = p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15.$$

Problem 2. Let $p_{\mathrm{d}}(n)$ be the number of partitions of n into distinct parts, and $p_{\mathrm{odd}}(n)$ the number of partitions of n into odd parts[1]. Prove *Euler's partition identity*

$$p_{\mathrm{d}}(n) = p_{\mathrm{odd}}(n),$$

by showing the generating function identities

$$\sum_{n=0}^{\infty} p_{\mathrm{d}}(n) x^n = \prod_{n=1}^{\infty} (1 + x^n),$$
$$\sum_{n=0}^{\infty} p_{\mathrm{odd}}(n) x^n = \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}}.$$

Solution 2. We multiply out,

$$\prod_{n=1}^{\infty} (1 + x^n) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \cdots$$

to see that the coefficient at x^n gets a contribution $+1$ from each product of monomials of the form $x^{d_1} x^{d_2} \cdots x^{d_k}$ where $d_1 + d_2 + \cdots + d_k = n$ and $0 < d_1 < d_2 < \cdots < d_k$. The possible

[1] As usual, we put $p_{\mathrm{d}}(0) = p_{\mathrm{odd}}(0) = 1$.

tuples $(d_1, \dots, d_k)$ represent the partitions of n into distinct parts, so the coefficient at x^n equals $p_{\mathrm{d}}(n)$.

Using the geometric series, we have

$$\prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = \prod_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} x^{k(2n-1)} \right)$$
$$= (1 + x^{1\cdot 1} + x^{2\cdot 1} + x^{3\cdot 1} + \dots)(1 + x^{1\cdot 3} + x^{2\cdot 3} + \dots)(1 + x^{1\cdot 5} + x^{2\cdot 5} + \dots) \cdots .$$

Hence, we get a contribtion $+1$ to x^n from products of the form $x^{k_1\cdot 1}x^{k_3\cdot 3}x^{k_5\cdot 5}\dots x^{k_{2j-1}\cdot(2j-1)}$ where

$$k_1 \cdot 1 + k_3 \cdot 3 + k_5 \cdot 5 + \dots + k_{2j-1} \cdot (2j-1) = n.$$

Such a tuple $(k_1, k_3, k_5, \dots)$ corresponds to the partition of n into odd parts

$$n = \underbrace{1 + \dots + 1}_{k_1 \text{ times}} + \underbrace{3 + \dots + 3}_{k_3 \text{ times}} + \dots$$

Hence, the coefficient at x^n equals $p_{\mathrm{odd}}(n)$.

We can now compute

$$\begin{aligned} \sum_{n=0}^{\infty} p_{\mathrm{d}}(n) x^n &= \prod_{n=1}^{\infty} (1 + x^n) \\ &= \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} \\ &= \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} \\ &= \sum_{n=0}^{\infty} p_{\mathrm{odd}}(n) x^n, \end{aligned}$$

which implies $p_{\mathrm{d}}(n) = p_{\mathrm{odd}}(n)$.

Problem 3. Let $\sigma(n) = \sum_{d|n} d$ be the sum of the divisors of n. Show that its generating function is given by

$$\sum_{n=1}^{\infty} \sigma(n) x^n = \sum_{n=1}^{\infty} \frac{n x^n}{1-x^n}.$$

Solution 3. We compute

$$\begin{aligned}
\sum_{n=1}^{\infty} \sigma(n) x^n &= \sum_{n=1}^{\infty} \sum_{d|n} d x^n \\
&= \sum_{a=1}^{\infty} \sum_{d=1}^{\infty} d x^{ad} \qquad (a = n/d) \\
&= \sum_{d=1}^{\infty} d \left(\sum_{a=1}^{\infty} (x^d)^a \right) \\
&= \sum_{d=1}^{\infty} d \cdot \left(\frac{1}{1-x^d} - 1 \right) \\
&= \sum_{d=1}^{\infty} d \cdot \frac{x^d}{1-x^d}.
\end{aligned}$$

This finishes the proof.

Problem 4. The Fibonacci numbers F_m are defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_m = F_{m-1} + F_{m-2}.$$

For each $n \in \{1, \ldots, 15\}$, count the number of partitions of n into distinct non-consecutive Fibonacci numbers. Make a conjecture based on your results[2].

Solution 4. The first few Fibonacci numbers are given by $1, 1, 2, 3, 5, 8, 13, 21$. We make a table with the partitions of n into distinct non-consecutive Fibonacci numbers.

n	partitions
1	1
2	2
3	3
4	$3+1$
5	5
6	$5+1$
7	$5+2$
8	8
9	$8+1$
10	$8+2$
11	$8+3$
12	$8+3+1$
13	13
14	$13+1$
15	$13+2$

In each case, the number of partitions is precisely 1, so one might conjecture that this is always the case. Indeed, Zeckendorf's Theorem states that every natural number n can be written in a unique way as a sum of distinct, non-consecutive Fibonacci numbers.

[2]Look up Zeckendorf's Theorem to validate your conjecture.

Problem 5 (sage)**.** Write a program that computes $p(n)$ using Euler's recursion. Use it to compute $p(100)$.