Elementary Number Theory - Exercise 2a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1.

1. Show that
$$\prod_{m+1<p\leq 2m+1} p \leq \binom{2m+1}{m}.$$
Hint: Which primes appear in the numerator and denominator of $\binom{2m+1}{m}$?

2. Show that
$$\binom{2m+1}{m} \leq 2^{2m}.$$
Hint: Rewrite 2^{2m+1} using the Binomial Theorem.

Solution 1. 1. Note that
$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$$
is an integer. Moreover, the primes $m+1 < p \leq 2m+1$ are all factors of the numerator $(2m+1)!$, but not of the denominator $m!(m+1)!$. This means that $\prod_{m+1<p\leq 2m+1} p$ divides $\binom{2m+1}{m}$, which implies the estimate.

2. Note that $\binom{2m+1}{m} = \binom{2m+1}{m+1}$. Using the Binomial Theorem, we find
$$2^{2m+1} = (1+1)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} \geq \binom{2m+1}{m} + \binom{2m+1}{m+1} = 2\binom{2m+1}{m}.$$

Problem 2. Show that the multiplicity of p in $n!$ is
$$\sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Solution 2. Out of the factors $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ of $n!$, precisely $\lfloor \frac{n}{p} \rfloor$ are divisible by p. This gives $\lfloor \frac{n}{p} \rfloor$ factors of p in $n!$. Similarly, $\lfloor \frac{n}{p^2} \rfloor$ of the factors $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ are divisible by p^2, which gives $\lfloor \frac{n}{p^2} \rfloor$ factors of p (although these factors are divisible by p^2, they only give a factor p each, since the second factor p was already accounted for).

Problem 3. Let $n \geq 3$. Show that, for $\frac{2}{3}n < p \leq n$, we have $p \nmid \binom{2n}{n}$.
Hint: How often does p appear in the numerator and denominator of $\binom{2n}{n}$?

Solution 3. Recall that
$$\binom{2n}{n} = \frac{(2n)!}{n!n!}.$$
Now for $\frac{2}{3} < p \leq n$ we have $3p > 2n$, which implies that p and $2p$ are the only multiples of p that appear in the numerator $(2n)!$, but p also appears in both $n!$ in the denominator. Hence, all factors of p cancel out. (Here we used that $n \geq 3$: then $\frac{2}{3}n < p$ implies $p \geq 3$. For $p = 2$ we have seen above that p and $2p = p^2$ divide the numerator, i.e. p appears *three* times in the numerator for $p = 2$. Indeed, for $n = 2$ we have $\binom{4}{2} = 6$, which is divisible by $p = 2$, and $\frac{2}{3} \cdot 2 < 2 \leq 2$, so the claim is not true for $n = 2$).

Problem 4. Show that

$$\binom{2n}{n} \geq \frac{2^{2n}}{2n}.$$

Hint: Binomial Theorem.

Solution 4. By the Binomial Theorem, we have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}$$

Now $\binom{2n}{n}$ is the middle summand, which is the biggest one of the $2n+1$ binomials appearing in the sum, so we can estimate

$$\sum_{k=0}^{2n} \binom{2n}{k} \leq (2n+1)\binom{2n}{n} \leq 2n\binom{2n}{n}.$$

This gives the desired estimate.

Problem 5 (sage)**.** Convince yourself numerically that we have

$$4^{\frac{1}{3}n} > (2n)^{1+\sqrt{2n}}$$

for $n > 4000$. For example, you could plot both functions. You could also try wolframalpha.com for this task! What would be a better threshold than 4000?