Elementary Number Theory - Exercise 2b

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Problem 1. Determine $\pi(100)$.

Solution 1. Writing out all primes ≤ 100 (e.g. using the Sieve of Eratosthenes), we find 25 primes, so $\pi(100) = 25$.

Problem 2. Show that $(3, 5, 7)$ is the only triple of primes of the form $(n, n+2, n+4)$.

Solution 2. Let $(n, n+2, n+4)$ be a sequence of three consecutive odd numbers, with $n > 3$. We show that one of them is divisible by 3, so they cannot all be prime. Note that n must be of the form $3k, 3k+1$, or $3k+2$, for some $k \in \mathbb{Z}$. If $n = 3k$, then n is divisible by 3, if $n = 3k+1$, then $n+2 = 3k+3$ is divisible by 3, and if $n = 3k+2$, then $n+4 = 3k+6$ is divisible by 3.

Problem 3. Show that, if $p > 3$ and $q = p+2$ are twin primes, then $p+q$ is divisible by 12.

Solution 3. We show that $p+q$ is divisible by 4 and by 3. Let us write $p = 2m+1$ and $q = 2m+3$. First, using $q = p+2$ we have

$$p + q = 2p + 2 = 2(2m+1) + 2 = 4m + 4,$$

which is divisible by 4. We can also write this as

$$p + q = 3m + 3 + m + 1,$$

so in order to show that $p+q$ is divisible by 3, we need to show that $m+1$ is divisible by 3. This is eqivalent to saying that m leaves remainder 2 when divided by 3. Indeed, the only other possible remainders are 0 and 1, in which cases $q = 2m+3$ or $p = 2m+1$ would be divisible by 3, respectively. This shows that $p+q$ is divisible by 3.

Problem 4. Show that there are infinitely many primes whose leading digit is 1.
Hint: How do intervals containing only numbers with leading digit 1 look like?

Solution 4. We are looking for primes in intervals of the form $(10^\ell, 2 \cdot 10^\ell)$. By Bertrand's Postulate, for any n there exits a prime with $n < p < 2n$. We apply this to $n = 10^\ell$ for $\ell = 1, 2, 3, \ldots$, so for each $\ell \in \mathbb{N}$ we obtain a prime p with

$$10^\ell < p < 2 \cdot 10^\ell.$$

Each such prime has leading digit 1.

Problem 5. Let $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$ be the set of primes. Show that every natural number $n > 6$ can be written as a sum of distinct elements from $\mathbb{P} \cup \{1\}$.
Hint: Use Bertrand's Postulate.

Solution 5. Let $m = \lfloor n/2 \rfloor$. By Bertrand's Postulate, there is a prime p between m and $2m$, and $n - p$ is less or equal to m. If $n - p = 1$ then we are done, otherwise we continue like this.

Problem 6 (Homework)**.**

1. Using the Prime Number Theorem, show the following strengthening of Bertrand's Postulate: For any $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a prime p with $n < p \leq (1+\varepsilon)n$.
2. Use the above to show that, for any given number m, there exist infinitely many primes starting with the digits of m. For example, there exist infinitely many primes starting with 313, e.g. $313, 3137, 31357, 313471, \ldots$.

Solution 6. 1. Note that for $x > 0$ the number of primes in the interval $(x, (1+\varepsilon)x]$ is given by

$$\pi((1+\varepsilon)x) - \pi(x),$$

so we want to show that this difference eventually becomes bigger than 1 for large x. By the Prime Number Theorem we have

$$\pi((1+\varepsilon)x) \sim \frac{(1+\varepsilon)x}{\log((1+\varepsilon)x)}$$

and

$$\pi(x) \sim \frac{x}{\log(x)}$$

for large x. Since

$$\log(x) \leq \log((1+\varepsilon)x) = \log(1+\varepsilon) + \log(x) \leq (1+\varepsilon/2)\log(x)$$

for large enough x, we have

$$\pi((1+\varepsilon)x) - \pi(x) \sim \left(\frac{1+\varepsilon}{1+\varepsilon/2} - 1\right)\frac{x}{\log(x)}.$$

Since the factor in front of $x/\log(x)$ is bigger than 0, and $x/\log(x)$ goes to ∞ as $x \to \infty$, we see that $\pi((1+\varepsilon)x) - \pi(x) \geq 1$ for large enough x.

2. Let us treat the example $m = 313$ to make the idea clearer. For general m it works in the same way. Since we are only interested in primes which start with 313, we should first look for a prime between 3130 and 3139, then for a prime between 31300 and 31399, then between 313000 and 313999, and so on. In general, we are looking for a prime between $313 \cdot 10^\ell$ and $313 \cdot 10^\ell + 10^\ell$. In order to apply the last item, we would like to bring the upper bound into the form $(1+\varepsilon) \cdot 313 \cdot 10^\ell$, so we should choose $\varepsilon = \frac{1}{1000}$ to obtain

$$(1+\varepsilon) \cdot 313 \cdot 10^\ell = (1 + \tfrac{1}{1000}) \cdot 313 \cdot 10^\ell = 313 \cdot 10^\ell + \frac{313}{1000} \cdot 10^\ell < 313 \cdot 10^\ell + 10^\ell.$$

Hence, with $\varepsilon = \frac{1}{1000}$, all numbers between $313 \cdot 10^\ell$ and $(1+\varepsilon) \cdot 313 \cdot 10^\ell$ will start with 313. Now the last item tells us that we can find a prime in such an interval for infinitely many large enough ℓ. This gives infinitely many primes starting with 313.

For a general starting sequence m with d digits, we choose $\varepsilon = 10^{-d}$ and see that all numbers between $m \cdot 10^\ell$ and $(1+\varepsilon) \cdot m \cdot 10^\ell$ start with m, so we obtain infinitely many primes starting with m.

Problem 7 (Homework)**.** Let p_n be the n-th prime. The n-th *primorial* $p_n\#$ is defined as the product of the primes up to p_n,

$$p_1\# = 2, \quad p_2\# = 2 \cdot 3 = 6, \quad p_3\# = 2 \cdot 3 \cdot 5 = 30, \quad p_4\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210, \quad \ldots$$

The n-th *Fortunate number*[1] F_n is the gap between $p_n\#$ and the next prime after $p_n\# + 1$. For example, the next prime after $p_2\# + 1 = 7$ is 11, so $F_2 = 11 - p_2\# = 11 - 6 = 5$.

Compute the first 10 Fortunate numbers[2]. Do you notice something special about them[3]?

[1] Named after the social anthropologist Reo Fortune.

[2] You might want to use sageMath or Wolframalpha for this

[3] You may look up *Fortune's Conjecture* to validate your findings

Solution 7. The first ten Fortunate numbers are

$$3, 5, 7, 13, 23, 17, 19, 23, 37, 61$$

The primorials get quite big very fast. For example, for $n = 10$ it is given by

$$p_{10}\# = 29\# = 6469693230.$$

It can be computed in Wolframalpha as `primorial(10)`. The next prime after $p_{10}\# + 1$ is

$$6469693291$$

which can be computed in Wolfamalpha as `next prime after 6469693231`. Hence, the 10-th Fortunate number is

$$6469693291 - 6469693230 = 61.$$

It appears that they are all prime. Fortunate's conjecture states that all Fortunate numbers should be prime, which is still open.

Problem 8 (sage)**.**

1. Implement the prime counting function $\pi(x)$.
2. Let $\psi(x)$ be the number of primes of the form $4k + 3$ that are $\leq x$. Implement $\psi(x)$ and make a guess what $\lim_{x\to\infty} \psi(x)/\pi(x)$ could be.
3. Write a program that prints a list of Fortunate numbers (bonus: get rid of duplicates and list the Fortunate numbers in ascending order), and check your conjecture from the last problem.