Elementary Number Theory - Exercise 3a

ETH Zürich - Dr. Markus Schwagenscheidt - Spring Term 2023

Problem 1. Let f be a multiplicative number-theoretic function which is not the constant 0-function. Show that $f(1) = 1$.

Solution 1. Suppose that $f(n) \neq 0$ for some $n \in \mathbb{N}$. Since f is multiplicative, we have $f(n) = f(n \cdot 1) = f(n)f(1) = 0$. Dividing by $f(n)$ shows $f(1) = 1$.

Problem 2.

1. Let $m, n \in \mathbb{N}$ be coprime. Show that every divisor $d \mid mn$ can be written uniquely as $d = d_1 d_2$ with $d_1 \mid m$ and $d_2 \mid n$.
2. Let $f \neq 0$ be a multiplicative number-theoretic function. Show that the summatory function of f,
$$F(n) = \sum_{d \mid n} f(d)$$
is multiplicative, as well.

Solution 2.

1. Since m, n are coprime, their prime factorizations
$$m = p_1^{\nu_1} \cdots p_r^{\nu_r}, \qquad n = q_1^{\mu_1} \cdots q_s^{\mu_s}$$
have no primes in common, that is, $p_j \neq q_k$ for all $1 \leq j \leq r$ and $1 \leq k \leq s$. Now every positive divisor $d \mid mn$ can be written uniquely as
$$d = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \cdot q_1^{\beta_1} \cdots q_s^{\beta_s}$$
with exponents $0 \leq \alpha_j \leq \mu_j$ and $0 \leq \beta_k \leq \nu_k$. Hence, d can be written uniquely as $d = d_1 d_2$ with
$$d_1 = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \mid m, \quad d_2 = q_1^{\beta_1} \cdots q_s^{\beta_s} \mid n.$$
2. Let m, n be coprime. We want to show that $F(mn) = F(n)$. By the last item, we have a bijection between the positive divisors $d \mid mn$ and the pairs (d_1, d_2) of positive divisors $d_1 \mid m$, $d_2 \mid n$, given by $d = d_1 d_2$. Hence, we can compute
$$\begin{aligned} F(mn) &= \sum_{d \mid mn} f(d) \\ &= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2) \\ &= \sum_{d_1 \mid m} f(d_1) \sum_{d_2 \mid n} f(d_2) \\ &= F(m)F(n). \end{aligned}$$
This shows that $F(n)$ is multiplicative.

Problem 3. We consider the divisor sum $\sigma_k(n) = \sum_{d|n} d^k$ for $k \in \mathbb{N}$.

1. Show that $\sigma_k(n)$ is multiplicative, but not completely multiplicative.
2. Let $n = p^m$ be a power of a prime p. Show that

$$\sigma_k(p^m) = \frac{p^{k(m+1)} - 1}{p^k - 1}.$$

3. Write down an explicit formula for $\sigma_k(n)$ in terms of the prime factorization of n.
4. Compute $\sigma_1(24)$ using the definition, and using your formula from the last item.

Solution 3.

1. Since $\sigma_k(n)$ is the summatory function of the multiplicative function $f(n) = n^k$, it is multiplicative by the last problem.

   We show that $\sigma_k(4) \neq \sigma_k(2)\sigma_k(2)$, so $\sigma_k(n)$ is not completely multiplicative. We have

$$\sigma_k(4) = 1 + 2^k + 4^k$$

   but

$$\sigma_k(2)\sigma_k(2) = (1 + 2^k)(1 + 2^k) = 1 + 2^k + 2^k + 4^k,$$

   and hence

$$\sigma_k(2)\sigma(2) - \sigma_k(4) = 2^k \neq 0.$$

2. The positive divisors of p^m are $1, p, p^2, \ldots, p^{m-1}, p^m$. Hence, we have

$$\sigma_k(p^m) = \sum_{j=0}^{m} (p^j)^k = \sum_{j=0}^{m} (p^k)^j = \frac{p^{k(m+1)} - 1}{p^k - 1},$$

   where used the formula for the geometric sum in the last step.

3. Since $\sigma_k(n)$ is multiplicative, we have

$$\sigma_k(n) = \prod_{j=1}^{s} \sigma_k(p^{\nu_j}) = \prod_{j=1}^{s} \frac{p^{k(\nu_j+1)} - 1}{p^k - 1},$$

   if $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$ is the prime factorization of n.

4. We have

$$\sigma_1(24) = 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60.$$

   Using $24 = 2^3 \cdot 3$ we have

$$\sigma_1(24) = \sigma_1(2^3)\sigma_1(3) = \frac{2^4 - 1}{2 - 1} \cdot \frac{3^2 - 1}{3 - 1} = 15 \cdot 4 = 60.$$

Problem 4. Let

$$\mu(n) = \begin{cases} (-1)^r, & \text{if } n \text{ is square-free, } n = p_1 \cdots p_r, \\ 0, & \text{otherwise,} \end{cases}$$

be the Möbius function. Check that μ is multiplicative, but not completely multiplicative, and show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hint: Use that $F(n) = \sum_{d|n} \mu(d)$ is multiplicative, and reduce the problem to the prime powers dividing n.

Solution 4. Let m, n be coprime. If some prime appears more than once in m or n, then it appears more than once in the product mn, and we have $\mu(mn) = 0$ and $\mu(m)\mu(n) = 0$. Otherwise, let $m = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$, with pairwise different primes $p_1, \dots, q_s$. Then $mn = p_1 \cdots p_r \cdot q_1 \cdots q_s$ is a product of $r + s$ different primes, so we get $\mu(mn) = (-1)^{r+s} = (-1)^r(-1)^s = \mu(m)\mu(n)$. This shows that μ is multiplicative.

Since $\mu(4) = 0$ but $\mu(2) \cdot \mu(2) = (-1) \cdot (-1) = 1$, the Möbius function is not completely multiplicative.

We have $\mu(1) = 1$ by definition. Hence, for $n = 1$ we have $\sum_{d|n} \mu(d) = \mu(1) = 1$, as claimed.

Now let $n > 1$, and write

$$n = p_1^{\nu_1} \cdots p_r^{\nu_r}$$

for the prime factorization of n. Since μ is multiplicative, we see as in the last exercise that $\sum_{d|n} \mu(d)$ is multiplicative, as well. Hence we can write

$$\sum_{d|n} \mu(n) = \prod_{j=1}^{r} \left(\sum_{d|p^{\nu_j}} \mu(d) \right) = \prod_{j=1}^{r} \left(\underbrace{\mu(1)}_{1} + \underbrace{\mu(p)}_{-1} + \underbrace{\mu(p^2)}_{0} + \cdots + \underbrace{\mu(p^{\nu_j})}_{0} \right) = 0.$$

Problem 5. (Homework) The Liouville function is defined by $\lambda(1) = 1$ and

$$\lambda(n) = (-1)^{\nu_1 + \ldots + \nu_r}, \quad \text{for } n = p_1^{\nu_1} \cdots p_r^{\nu_r}.$$

Check that λ is completely multiplicative and show that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square,} \\ 0, & \text{otherwise} \end{cases}$$

Hint: First show the identity for prime powers n, and then use multiplicativity.

Solution 5.

1. Let $m = p_1^{\nu_1} \cdots p_r^{\nu_r}$ and $n = q_1^{\mu_1} \cdots q_s^{\mu_s}$ (here, the p_j, q_k are not necessarily pairwise different). Then

   $$mn = p_1^{\nu_1} \cdots p_r^{\nu_r} \cdot q_1^{\mu_1} \cdots q_s^{\mu_s}$$

   (again, the prime powers need not be pairwise different here), so

   $$\lambda(mn) = (-1)^{\nu_1 + \ldots + \nu_r + \mu_1 + \ldots + \mu_s} = \lambda(m)\lambda(n).$$

2. We first compute $\sum_{d|n} \lambda(n)$ for prime powers $n = p^m$. We have

$$\sum_{d|p^m} \lambda(d) = \sum_{j=0}^{m} \lambda(p^j) = \sum_{j=0}^{m} (-1)^j = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Since λ is multiplicative, its summatory function is multiplicative. Hence

$$\sum_{d|n} \lambda(d) = \prod_{j=0}^{s} \left(\sum_{d|p_j^{\nu_j}} \lambda(d) \right) = \begin{cases} 1, & \text{if all } \nu_j \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

Finally, n is a square if and only if all the exponents ν_j in its prime factorization are even, which gives the stated formula.

Problem 6 (sage)**.** Implement the following functions in sage:

1. `sigma(k,n)`: the divisor sum $\sigma_k(n)$.
2. `moebius(n)`: the Moebius function $\mu(n)$.
3. `phi(n)`: Euler's totient function $\varphi(n)$.