Elementary Number Theory - Exercise 4b

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Problem 1. Show that

$$\sum_{d|n} \sigma\left(\frac{n}{d}\right)\varphi(d) = n\tau(n).$$

Solution 1. The left-hand side is the convolution

$$\sum_{d|n} \sigma\left(\frac{n}{d}\right)\varphi(d) = \sigma * \varphi.$$

By definition, we have $\sigma = \mathbf{1} * \mathrm{id}$. Moreover, we have seen in an exercise problem above that $\varphi = \mu * \mathrm{id}$. Hence we get

$$\sigma * \varphi = (\mathbf{1} * \mathrm{id}) * (\mu * \mathrm{id}) = (\mathbf{1} * \mu) * (\mathrm{id} * \mathrm{id}),$$

where we used that convolution is associative and commutative. We know that $\mathbf{1} * \mu = e$. Moreover, we have $(\mathrm{id} * \mathrm{id})(n) = \sum_{d|n} d \cdot \frac{n}{d} = n \sum_{d|n} 1 = n\tau(n)$. Taking everything together, we obtain the stated formula.

Problem 2. Recall Liouville's λ-function defined by $\lambda(1) = 1$ and

$$\lambda(n) = (-1)^{\nu_1 + \ldots + \nu_r}, \quad \text{for } n = p_1^{\nu_1} \cdots p_r^{\nu_r}.$$

Show that the Dirichlet-inverse of λ is given by $\hat{\lambda}(n) = |\mu(n)|$.

Solution 2. Since λ is completely multiplicative, its inverse is given by

$$\hat{\lambda}(n) = \mu(n)\lambda(n).$$

The right-hand side vanishes whenever n is not square-free. However, if $n = p_1 \cdots p_r$ is square-free, then $\lambda(n) = \lambda(p_1) \cdot \lambda(p_r) = (-1)^r = \mu(n)$. Hence, we get

$$\hat{\lambda}(n) = \mu(n)\lambda(n) = |\mu(n)|.$$

Problem 3. Show that $\tau^3 * \mathbf{1} = (\tau * \mathbf{1})^2$.

Solution 3. Since τ (and hence τ^3) and $\mathbf{1}$ are multiplicative, both $\tau^3 * \mathbf{1}$ and $(\tau * \mathbf{1})^2$ are multiplicative. Hence, it suffices to prove the above identity on prime powers p^m. We have

$$(\tau^3 * \mathbf{1})(p^m) = \sum_{d|p^m} \tau^3(d) = \sum_{j=0}^{m} \tau^3(p^j) = \sum_{j=0}^{m} (j+1)^3,$$

where we used that $\tau(p^j) = j + 1$. On the other hand, we have

$$(\tau * \mathbf{1})^2(p^m) = \left(\sum_{d|p^m} \tau(d)\right)^2 = \left(\sum_{j=0}^{m} \tau(p^j)\right)^2 = \left(\sum_{j=0}^{m} (j+1)\right)^2.$$

Hence, the claimed identity boils down to

$$\sum_{j=0}^{m}(j+1)^3 = \left(\sum_{j=0}^{m}(j+1)\right)^2 .$$

This can be easily proved by induction on m.

Problem 4. Show that

$$\tau(mn) = \sum_{d|\gcd(m,n)} \mu(d)\tau(m/d)\tau(n/d)$$

for all $m, n \in \mathbb{N}$.

Solution 4. We first show that we can assume that m, n have the same prime divisors, that is, $p \mid m$ if and only if $p \mid n$ for any prime p. Indeed, by splitting m, n into a product of prime powers, we can uniquely write $m = m_1 m_2$ and $n = n_1 n_2$ where m_1, n_1 have the same prime divisors, and $\gcd(m_2, n_2) = \gcd(m_1, m_2) = \gcd(n_1, n_2) = 1$. Since τ is multiplicative, we have

$$\tau(mn) = \tau(m_1 n_1 m_2 n_2) = \tau(m_1 n_1)\tau(m_2)\tau(n_2),$$

and, since $\gcd(m, n) = \gcd(m_1, n_1)$,

$$\sum_{d|\gcd(m,n)} \mu(d)\tau(m/d)\tau(n/d) = \tau(m_2)\tau(n_2) \sum_{d|\gcd(m_1,n_1)} \mu(d)\tau(m_1/d)\tau(n_1/d).$$

We see that $\tau(m_2)\tau(n_2)$ appear on both sides and can be cancelled out. Hence, it remains to prove the claimed identity in the case that m, n have the same prime divisors.

Write

$$m = p_1^{\mu_1} \cdots p_r^{\mu_r}, \quad n = p_1^{\nu_1} \cdots p_r^{\nu_r},$$

with distinct primes $p_1, \ldots, p_r$, and $\mu_j, \nu_j \in \mathbb{N}$. Since τ is multiplicative and $\tau(p^m) = m + 1$, we have

$$\tau(mn) = \prod_{j=1}^{r}(\mu_j + \nu_j + 1).$$

On the other hand, since $\mu(d) = 0$ if d is contains some prime factor p_j more than once, the sum over $d \mid \gcd(m, n)$ can be written as a sum over $d = p_1^{e_1} \cdots p_r^{e_r}$ where $e_j \in \{0, 1\}$. Hence, we can compute

$$\begin{aligned}
\sum_{d|\gcd(m,n)} \mu(d)\tau(m/d)\tau(n/d) &= \sum_{e_1=0}^{1} \cdots \sum_{e_r=0}^{1} \mu(p_1^{e_1} \cdots p_r^{e_r})\tau(p_1^{\mu_1-e_1} \cdots p_r^{\mu_r-e_r})\tau(p_1^{\nu_1-e_r} \cdots p_r^{\nu_r-e_r}) \\
&= \sum_{e_1=0}^{1} \cdots \sum_{e_r=0}^{1} \mu(p_1^{e_1}) \cdots \mu(p_r^{e_r})\tau(p_1^{\mu_1-e_1}) \cdots \tau(p_r^{\mu_r-e_r})\tau(p_1^{\nu_1-e_r}) \cdots \tau(p_r^{\nu_r-e_r}) \\
&= \prod_{j=1}^{r} \left(\sum_{e_j=0}^{1} \mu(p_j^{e_j})\tau(p^{\mu_j-e_j})\tau(p^{\nu_j-e_j}) \right) \\
&= \prod_{j=1}^{r} \left(\tau(p^{\mu_j})\tau(p^{\nu_j}) - \tau(p^{\mu_j-1})\tau(p^{\nu_j-1}) \right) \\
&= \prod_{j=1}^{r} ((\mu_j+1)(\nu_j+1) - (\mu_j\nu_j)) \\
&= \prod_{j=1}^{r} (\mu_j+\nu_j+1) \\
&= \tau(mn).
\end{aligned}$$

This shows the claimed identity.

Problem 5. Show that every even perfect number $n > 6$ can be written as a sum of the first d odd cubes, $n = 1^3 + 3^3 + 5^3 + \ldots + d^3$, for a suitable odd $d \in \mathbb{N}$. For example, $28 = 1^3 + 3^3$. *Hint:* Use the summation formula $1^3 + 2^3 + 3^3 + 4^3 + \ldots + (d-1)^3 + d^3 = \frac{d^2(d+1)^2}{4}$.

Solution 5. Let d be an odd natural number. We split the sum over *all* cubes $1^3+2^3+\ldots+d^3$ up to d^3 into the even and the odd cubes, and write

$$\begin{aligned}
\frac{d^2(d+1)^2}{4} &= \sum_{k=0}^{d} k^3 = \sum_{\substack{k=0\\k \text{ odd}}}^{d} k^3 + \sum_{\substack{k=0\\k \text{ even}}}^{d} k^3 = \sum_{\substack{k=0\\k \text{ odd}}}^{d} k^3 + 2^3 \sum_{k=0}^{(d-1)/2} k^3 \\
&= \sum_{\substack{k=0\\k \text{ odd}}}^{d} k^3 + 8 \cdot \frac{(\frac{d-1}{2})^2(\frac{d+1}{2})^2}{4},
\end{aligned}$$

so we obtain the formula

$$\begin{aligned}
\sum_{\substack{k=0\\k \text{ odd}}}^{d} k^3 &= \frac{d^2(d+1)^2}{4} - \frac{(d-1)^2(d+1)^2}{8} \\
&= \frac{(2d^2 - (d-1)^2)(d+1)^2}{8} \\
&= \frac{((d+1)^2 - 2)(d+1)^2}{8}. \\
&= \left(\frac{(d+1)^2}{2} - 1 \right) \cdot \frac{(d+1)^2}{4}.
\end{aligned}$$

Let n be an even perfect number. By Euler's criterion, it is of the form $n = 2^{m-1}(2^m - 1)$ for some m such that $2^m - 1$ is prime. We have seen that this is only possible if m is prime. Since we excluded $n = 6$ (which corresponds to $m = 2$), we can assume that m is odd. Looking back at the formula for the sum $1^3 + 3^3 + 5^3 + \ldots + d^3$ that we obtained above, we see that want to choose d such that

$$2^m = \frac{(d+1)^2}{2},$$

that is, $d = 2^{\frac{m+1}{2}} - 1$. Since we assumed that m is odd, this is possible.

Let's do a quick check: For $m = 3$ we have $d = 2^{\frac{3+1}{2}} - 1 = 3$, and indeed we have $28 = 1^3 + 3^3$. For $m = 5$ we have $d = 2^{\frac{5+1}{2}} - 1 = 7$, and

$$1^3 + 3^3 + 5^3 + 7^3 = 496 = 2^4(2^5 - 1),$$

which is the third even perfect number.