Heights

Pia Herkenrath Ana Pavlaković

ETH Zürich

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Table of Contents

Height of A Rational Number

Definition

Let $x = \frac{m}{n}$ be a rational number in lowest terms. The *height* of x is given by

$$H(x) = H\left(\frac{m}{n}\right) = \max\{|m|, |n|\}$$

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Finiteness Property

The set of all rational numbers with height smaller than a fixed constant d is finite.

Height of A Point on A Cubic Curve

Consider a non-singular cubic curve C:

$$y^2 = f(x) = x^3 + ax^2 + bx + c$$

where a, b, and c are integer coefficients.
Let $P = (x, y)$ be a rational point on the curve.

Definition

The height of P is defined as

$$H(P) = H(x)$$

Convention: $H(\mathcal{O}) = 1$

Height of A Point on A Cubic Curve

Definition

The small h height of P is defined as

$$h(P) = logH(P)$$

Results for Proving Mordell's Theorem

Lemma 1

For every real number M the set

$$\{P \in C(\mathbb{Q}) : h(P) \leq M\}$$

is finite.

Results for Proving Mordell's Theorem

Lemma 2

Let $P_0 \in C(\mathbb{Q})$ be fixed. Then there exists a constant κ_0 depending on P_0 and on a, b, and c such that

$$h(P + P_0) \leq 2h(P) + \kappa_0$$

for every point $P \in C(\mathbb{Q})$.

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Lemma 3

There exists a constant κ depending on a, b, and c such that

$$h(2P) \geq 4h(P) - \kappa$$

for all $P \in C(\mathbb{Q})$.

Results for Proving Mordell's Theorem

Lemma 4

The index $(C(\mathbb{Q}) : 2C(\mathbb{Q}))$ is finite.

Preliminary Results

Remark 1

Let $P = (x, y)$ be a rational point on the curve. Then x and y have the form

$$x = \frac{m}{e^2} \qquad y = \frac{n}{e^3}$$

for $m, n, e \in \mathbb{Z}$, with $e > 0$ and $\gcd(m, e) = \gcd(n, e) = 1$.

Preliminary Results

Remark 2

Let the point $P = \left(\frac{m}{e^2}, \frac{n}{e^3}\right)$ be given in lowest terms. Then the following inequalities hold:

$$|m| \leq H(P) \qquad e \leq H(P)^{1/2} \qquad |n| \leq KH(P)^{3/2}$$

where $K > 0$ is a constant depending on a, b, and c.

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For Further Reading

J. Silverman and J. Tate.
Rational Points on Elliptic Curves.
Springer, 1992.