# Modular Forms

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## 1 The $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{H}$

Earlier in this course, when we talked about Weierstrass $\wp$-function, we have seen the notion of a lattice $\Omega$ in the complex plane $\mathbb{C}$, and associated certain values to it, for example the Eisenstein series $G_k(\Omega)$. The study of this dependence on the lattice leads in a natural way to the definition of the modular group action on the complex upper half plane and consequently to modular forms and functions, as we will explain in this section.

We represent a lattice $\Omega$ by a pair of two $\mathbb{R}$-linearly independent vectors $(\omega_1, \omega_2) \in \mathbb{C}^2$, such that $\Omega = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. The group $\mathrm{GL}_2(\mathbb{R})$ acts $\mathbb{R}$-linearly on the set of these linearly independent pairs by

$$M(\omega_1, \omega_2) = (a\omega_1 + b\omega_2, c\omega_1 + d\omega_2), \quad \text{where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

Now notice that for some $M \in \mathrm{GL}_2(\mathbb{R})$, the two vectors $M(\omega_1, \omega_2)$ lie again in $\Omega$ if and only if $M$ has entries in $\mathbb{Z}$. In this case, the lattice spanned by $M(\omega_1, \omega_2)$ is equal to $\Omega$ if and only if $\det(M) = ad - bc = \pm 1$. If we furthermore assume all these bases to have some fixed orientation, then the last condition turns into $\det(M) = 1$. Thus, two positively (resp. negatively) oriented pairs of vectors $(\omega_1, \omega_2)$, $(\eta_1, \eta_2)$ span the same lattice if and only if $(\eta_1, \eta_2) = M(\omega_1, \omega_2)$ for some $M \in \mathrm{SL}_2(\mathbb{Z}) = \{A \in \mathbb{Z}^{2\times 2} | \det(A) = 1\}$. We call $\mathrm{SL}_2(\mathbb{Z})$ the *modular group*. A function of $(\omega_1, \omega_2)$ that only depends on the geometric locus of the lattice spanned by the two vectors, will therefore be invariant under this $\mathrm{SL}_2(\mathbb{Z})$-action.

To make this description more efficient, we now parametrise the set of lattices only in terms of one complex parameter $\tau$, instead of two basis vectors. For this, we will identify different bases that are complex multiples of each other. This allows us to assume that our basis is negatively oriented and $\omega_2 = 1$, $\omega_1 = \tau \in \mathbb{H} = \{z \in \mathbb{C} | \Im z > 0\}$. Using this identification, $\mathrm{SL}_2(\mathbb{Z})$ transforms our parameter $\tau$ in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, 1) = (a\tau + b, c\tau + d) \sim \left( \frac{a\tau + b}{c\tau + d}, 1 \right).$$

Since our matrix has positive determinant, this new basis is again negatively oriented, so $\frac{a\tau+b}{c\tau+d} \in \mathbb{H}$. Hence, this describes an $\mathrm{SL}_2(\mathbb{Z})$-action on the upper half plane $\mathbb{H}$ through Möbius transformations, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

We will note a few properties on the modular group and its action on the upper half plane without proof. First, $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the first of which corresponds to a translation $\tau \mapsto \tau + 1$ and the second to $\tau \mapsto -\frac{1}{\tau}$, which is geometrically an inversion on the unit circle, followed by a reflection along the imaginary axis.

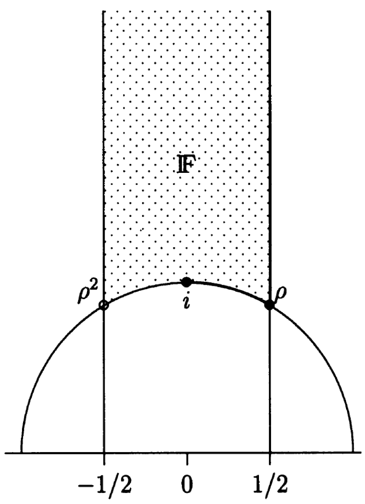
Secondly, we will describe a fundamental domain of this action, i.e. a subset $\mathbb{F} \subset \mathbb{H}$ such that every orbit under the $\mathrm{SL}_2(\mathbb{Z})$-action has exactly one element in $\mathbb{F}$. Such a fundamental domain is given by

$$\mathbb{F} = \left\{ \tau \in \mathbb{H} \;\middle|\; -\frac{1}{2} < \Re(\tau) \leq \frac{1}{2}, |\tau| \geq 1, \text{ and } |\tau| > 1 \text{ for } \Re(\tau) < 0 \right\}.$$

A drawing of the fundamental domain is shown on the right. Here, the bottom right corner point is

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

and the bottom left is $\rho^2 = \rho - 1$.




Third, we look at the fixed points of the action in $\mathbb{F}$. During the following, we will write $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For every point $\tau \in \mathbb{H}$, denote by $\Gamma_\tau = \{M \in \Gamma | M\tau = \tau\}$ the stabilizer subgroup of $\tau$. One can show that $\Gamma_i = \{\pm I, \pm J\}$, where $I$ denotes the identity, $\Gamma_\rho = \{\pm I, \pm U, \pm U^2\}$, where $U = TJ$, and $\Gamma_\tau = \{\pm I\}$ for all other $\tau \in \mathbb{F}$. The relation $\Gamma_{M\tau} = M\Gamma_\tau M^{-1}$ now gives us all stabilizer subgroups of points in $\mathbb{H}$. We denote the space of orbits by $\Gamma \backslash \mathbb{H} = \{\Gamma\tau | \tau \in \mathbb{H}\}$, whose elements correspond bijectively with their representatives in $\mathbb{F}$.

Now let us go back to the study of functions $f(\omega_1, \omega_2)$ invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. For Eisenstein Series we have observed a law that looks like $f(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} f(\omega_1, \omega_2)$, $k \in \mathbb{Z}$, where $\lambda \in \mathbb{C}$, geometrically, is a successive rotation and stretching of the lattice. When passing to our parametrization by the single variable $\tau \in \mathbb{H}$, such function $f$ transforms as follows:

$$\begin{aligned} f(\tau, 1) &= f(M(\tau, 1)) \\ &= f(a\tau + b, c\tau + d) \\ &= (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}, 1\right) \\ &= (c\tau + d)^{-k} f(M\tau, 1) \end{aligned}$$

Writing $f(\tau, 1) = \tilde{f}(\tau)$, this gives us the transformation law

$$\tilde{f}(M\tau) = (c\tau + d)^k \tilde{f}(\tau).$$

This formula will serve as our motivation to define modular forms.

# 2 Modular Forms

## 2.1 Preliminary remarks

As elaborated above it is natural to consider the following $\mathrm{SL}_2(\mathbb{Z})$ action on the space of meromorphic functions $K(\mathbb{H})$

$$(f|_k M)\tau = (c+d)^{-k} f(M\tau)$$

for $f \in K(\mathbb{H})$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and a fixed $k \in \mathbb{Z}$. It is an easy exercise to check that this is indeed a group action, i.e.

$$f|_k M_k N = f|MN$$

for all $M, N \in \mathrm{SL}_2(\mathbb{Z})$.

More specifically we will be interested in meromorphic functions on the upper half plane that are invariant under this group action (again for a fixed $k \in \mathbb{Z}$), that is in those $f \in K(\mathbb{H})$ such that

$$f = f|_k M$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$. We will call them modular invariant in what follows.

Before we explain how this modular invariance and the meromorphic property interact do we want to give an easy consequence of the invariance:

**Lemma.** *For a given $k \in \mathbb{Z}$, $f \in K(\mathbb{H})$ is modular invariant in the above sense if and only if $f(\tau+1) = f(\tau)$ and $f(-\frac{1}{\tau}) = \tau^k f(\tau)$ for all $\tau \in \mathbb{H}$.*

*Proof.* This is a consequence of the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ and the fact that the operation is a group action. □

Functions that are modular invariant are by definition determined by their values on the fundamental domain of the $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. In the following we will thus think about such a function via its restriction to the fundamental domain. Since the set of poles of a meromorphic function has to be discrete, there are the following possibilities for the behavior of a modular invariant function on $\mathbb{H}$:

1. There can be infinitely many poles in imaginary directions of the fundamental domain strip.

2. There could be only finitely many poles, so that the function is holomorphic for large enough imaginary coordinate values, but still unbounded.

3. The function could be bounded for large enough coordinate values, but not going to zero

4. The function could go to zero for imaginary coordinate value going to infinity.

A different way of thinking about this is as follows. Consider the map

$$\mathbb{H} \to D_1 \setminus \{0\},\ z \mapsto e^{2\pi i z}$$

which maps the upper half plane surjectively with periodicity one to the disk of radius one with the origin removed. Using this map we can compare one periodic functions on the upper half plane and functions on $D_1 \setminus \{0\}$. Thus we can apply this procedure to our modular invariant functions. A modular invariant function that falls into category $i$ in the classification above corresponds to a function on $D_1 \setminus \{0\}$ that falls in the following category: It is a meromorphic functions such that

1. the poles accumulate at zero, i.e. we have an essential singularity,
2. there is a non essential singularity at zero, i.e. a simple pole,
3. there is a removable singularity at zero, which is not a root,
4. there is a root at zero.

There is yet another way to describe this behavior. Since the modular invariant functions are especially periodic we have a Fourier series decomposition. Again corresponding to the cases before it looks like

1. $f(\tau) = \sum_{m\in\mathbb{Z}} a_{f,m} q^m$ with $q = e^{2\pi i \tau}$,
2. $\exists m_0 \in \mathbb{Z} : a_{f,m} = 0\ \forall m \leq m_0$,
3. $m_0 = 0$ for above statement,
4. $m_0 = 1$ for statement 2.

## 2.2 Definitions

Having analyzed this in detail we finally come to the definitions. We give modular invariant functions falling into categories 1.) till 4.) the following names:

**Definition.** *A modular function of weight $k$ is*

- *a meromorphic function on $\mathbb{H}$,*
- *modular invariant with integer $k$*
- *and falls into category 1.).*

**Definition.** *A modular form of weight $k$ is*

- *a meromorphic function on $\mathbb{H}$,*
- *modular invariant with integer $k$,*
- *and falls into category 2.), i.e , i.e has almost a pole at infinity (or equivalently a terminating Fourier expansion).*

We denote the space of modular forms of weight $k$ by $\mathbb{V}_k$.

**Definition.** *An entire modular form of weight $k$ is*

- *a meromorphic function on $\mathbb{H}$,*
- *modular invariant with integer $k$,*
- *and falls into category 3.), i.e. is bounded at infinity.*

We denote the space of entire modular forms of weight $k$ by $\mathbb{M}_k$.

**Definition.** *A cusp form of weight $k$ is*

- *a meromorphic function on $\mathbb{H}$,*
- *modular invariant with integer $k$,*
- *and falls into category 4.), is has a zero at infinity.*

We denote the space of modular forms of weight $k$ by $\mathbb{S}_k$.
We list some fundamental properties of those spaces before coming to examples.

**Theorem.** *The spaces $\mathbb{S}_k \subset \mathbb{M}_k \subset \mathbb{V}_k$ form respectively a graded algebra over $\mathbb{C}$ via*

$$\mathbb{S} = \bigoplus_k \mathbb{S}_k$$

$$\mathbb{V} = \bigoplus_k \mathbb{V}_k$$

$$\mathbb{M} = \bigoplus_k \mathbb{M}_k$$

*Proof.* These are easy consequences of the definitions and the properties of meromorphic functions. They imply that next to $f + g \in \mathbb{S}_k$ for $f, g \in \mathbb{S}_k$ also $f \cdot g \in \mathbb{S}_{k+l}$ for $f \in \mathbb{S}_k, g \in \mathbb{S}_l$, thus the graded algebra property. □

**Theorem.** *$\mathbb{V}$ is an algebra over the field $\mathbb{V}_0$.*

*Proof.* This statement is well defined and correct since for $f \in \mathbb{V}_k$ the meromorphic function $\frac{1}{f} \in \mathbb{V}_{-k}$. □

**Theorem.** *$\mathbb{V}_k = \{0\}$ for $k = 2m + 1$ and $m \in \mathbb{Z}$*

*Proof.* Plugging minus the identity matrix into the modular invariance formula immediately gives the result. □

# 3 Examples

As explained earlier by analyzing the Weierstrass $\wp$-function we are lead to the Eisenstein series. This is initially defined in dependence of a lattice, but we can normalize lattices in a certain way such that the Eisenstein series is a function depending on a parameter $\tau \in \mathbb{H}$.

### 3.1 The Eisenstein series

We recall that $G_k(\tau) = \sum_{m,n\in\mathbb{Z}\setminus\{0\}}(m\tau+n)^{-k}$. In an earlier talk we have seen that this is indeed a meromorphic function on the upper half plane. We show that $G_k$ is modular invariant with respect to the integer $k$.

$$(G_k|_kM)(\tau) = (c\tau+d)^{-k} \sum_{m,n\in\mathbb{Z}^2\setminus\{0\}} (m\frac{a\tau+b}{c\tau+d}+n)^{-k} = \sum_{m,n\in\mathbb{Z}^2\setminus\{0\}} (m(a\tau+d)+n(c\tau+d))^{-k}$$

$$= \sum_{m,n\in\mathbb{Z}^2\setminus\{0\}} (ma+nc)\tau+mb+nd^{-k} = \sum_{m,n\in\mathbb{Z}^2\setminus\{0\}} (m\tau+n)^{-k}$$

In the last step we have used that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\mathrm{SL}_2(\mathbb{Z})$ and thus induces a bijection of $\mathbb{Z}^2 \setminus \{0\}$ and that the series converges absolutely, so that we can rearrange its terms.
Lastly we sketch the Fourier decomposition of $G_k$:

$$G_k(\tau) = \sum_{n\neq 0} n^{-k} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{m\in\mathbb{Z}} (m\tau+n)^{-k} = 2\zeta(\tau) + \frac{2\pi ik}{(k-1)!} \sum_{n=0}^{\infty} \sigma_{k-1} q^n$$

with $q = e^{2\pi i\tau}$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{n-k}$. The first term is just by definition, whereas the second term requires some computation (and that $k \geq 0$) which we omit.

The upshot of these calculations and remarks is that $G_k \in M_k$.

### 3.2 The Delta Function

By definition the Delta Function $\Delta$ is given by the following polynomial in $G_k$

$$\Delta(\tau) = \frac{(\frac{G_4(\tau)}{2\zeta(4)})^3 - (\frac{(G_6(\tau)}{2\zeta(6)})^2}{1728}$$

We also directly give the first summands of the Fourier series:

$$\Delta(\tau) = q - 24q^2 + 252q^3 + \cdots$$

By the result about the Eisenstein series earlier and the Fourier series we immediatly see that $\Delta$ it is a cusp form of weight 12 since $3 \times 4 = 2 \times 6 = 12$.
As an aside we quote that $\Delta$ also has the following representation:

$$\Delta(\tau) = q\Pi_{n=1}^{\infty}(1-q^n)^{24}$$

which is interesting from a number theoretic viewpoint.

### 3.3 The $j$-invariant

The $j$-invariant is by definition given by

$$j(\tau) = \frac{G_4(\tau)^3}{2\Delta(\tau)\zeta(4)}$$

and we have seen its importance in an earlier talk about isomorphism classes of lattices.
From the Fourier expansion of $\Delta$ and $G_4$ it follows that

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots$$

and with these two results combined it follows that $j$ is a modular form of weight zero.

# 4 The graded algebra $\mathbb{M}$

This section focuses on algebraic properties of the graded, commutative algebra of entire modular forms, $\mathbb{M} = \bigoplus_k \mathbb{M}_k$, namely, dimension and generators. First, we want to look at low weights $k < 12$, for which the following complex-analytic result will be useful.

## 4.1 The weight formula

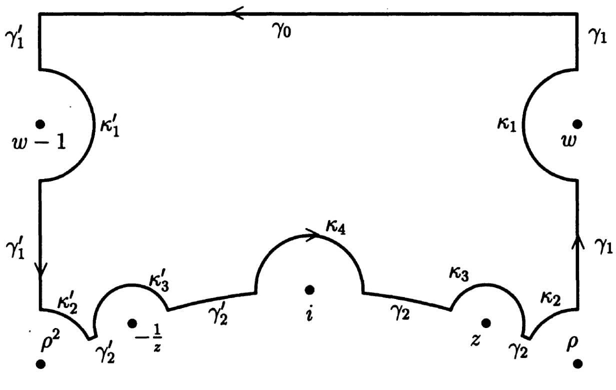
Let us denote $\mathbb{F}^* = \mathbb{F} \cup \{\infty\}$. Furthermore, we define for $\tau \in \mathbb{H}$, the order $\mathrm{ord}(\tau) = \frac{1}{2}\#\Gamma_\tau$, i.e. $\mathrm{ord}(\rho) = 3$, $\mathrm{ord}(i) = 2$ and $\mathrm{ord}(\omega) = 1$ for all other $\omega \in \mathbb{F}^*$. We will refer to the following result as the *weight formula*.

**Theorem.** *Any modular form $0 \neq f \in \mathbb{V}_k$ satisfies the equation*

$$\sum_{\omega \in \mathbb{F}^*} \frac{1}{\mathrm{ord}(\omega)} \mathrm{ord}_\omega(f) = \frac{k}{12}.$$

To prove this, one uses the argument principle in order to count the multiplicities of zeros and poles of $f$ through a path integral in $\mathbb{F}$. We will not go through the calculations needed for the proof here but briefly explain the argument. For more details, see [1], pages 146-149.

The path used in the argument looks as follows:




Here $\omega$ and $z$ are possible additional poles or zeros of $f$ on the boundary of $\mathbb{F}$, the $\kappa_\nu$ and $\kappa_\nu'$ are circular arcs with radius $\epsilon$ small enough so that the circular neighborhoods of their center do not contain any more zeros or poles. Also, we can pick $\gamma_0$ far enough towards $\infty$ such that there are no poles or zeros left above. The argument principle now tells us that

$$\int_\gamma F(\tau)d\tau = 2\pi i \sum_{\omega\in\mathbb{F}^\circ} \operatorname{ord}_\omega(f),$$

where $F(\tau) = \frac{f'(\tau)}{f(\tau)}$, $\mathbb{F}^\circ$ denotes the interior of $\mathbb{F}$, and $\gamma$ is the combined path running counter-clockwise.

Now we will look at all the components of the path integral seperately.

- Using the Fourier expansion of $F$ one can show that the integral along $\gamma_0$ evaluates to $-2\pi i \operatorname{ord}_\infty(f)$.
- Due to translation invariance of $F$, the integrals along $\gamma_1$ and $\gamma_1'$ cancel.
- The paths $\kappa_1$ and $\kappa_1'$, again due to translation invariance, sum to the $-2\pi i$ times the residue of $F$ at $\omega$, which gives $-2\pi i \operatorname{ord}_\omega(f)$. Due to $J$-invariance, the same can be said for the two paths $\kappa_3$ and $\kappa_3'$.
- Due to $J$-invariance, the integral along $\kappa_4$ evaluates to $-2\pi i$ times *half* the residue of $F$ at $i$, which gives $-\pi i \operatorname{ord}_i(f)$.
- Due to $U$- and $T$-invariance, the two integrals along $\kappa_2$ and $\kappa_2'$ combine to $-2\pi i$ times *one third* of the residue of $F$ at $\rho$, which gives $-\frac{2\pi i}{3}\operatorname{ord}_\rho(f)$.

If we combine all of these different terms and divide by $2\pi i$, we arrive at the weight formula.

## 4.2 Dimensions and Generators

The weight formula gives us powerful restrictions on entire modular functions. In order to see this, let us spell out the weight formula:

$$\frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \operatorname{ord}_\infty(f) + \sum_{\omega\in\mathbb{F}\setminus\{i,\rho\}} \operatorname{ord}_\omega(f) = \frac{k}{12}$$

For an entire modular form $f$, all the $\operatorname{ord}_\omega(f)$ are non-negative integers. For weight $k < 12$, this tells us that the zeros of $f$ have to be located at $i$ or $\rho$. The following table shows their possible zeros and orders.

| $k$ | $k/12$ | $\operatorname{ord}_\infty(f)$ | $\operatorname{ord}_i(f)$ | $\operatorname{ord}_\rho(f)$ | f has zeros at | of weight |
|---|---|---|---|---|---|---|
| 4 | 1/3 | 0 | 0 | 1 | $\rho$ | 1 |
| 6 | 1/2 | 0 | 1 | 0 | $i$ | 1 |
| 8 | 2/3 | 0 | 0 | 2 | $\rho$ | 2 |
| 10 | 5/6 | 0 | 1 | 1 | $\rho$ and $i$ | 1 |

This allows us to classify entire modular forms of low weight:

**Proposition.** *a)* $\mathbb{M}_k = \{0\}$ *for* $k < 0$.

*b)* $\mathbb{M}_0 = \mathbb{C}$.

*c)* $\mathbb{M}_2 = \{0\}$.

*d)* $\mathbb{M}_k = \mathbb{C}G_k = \mathbb{C}G_k^*$ *and* $\mathbb{S}_k = \{0\}$ *for* $k = 4, 6, 8, 10$.

*Proof.* a) For $k < 0$, the left hand side of the weight formula would be negative, which is not possible.

b) For $k = 0$, the weight formula tells us that entire forms of weight $k$ can not have any zeros. Consider $g = f - f(i)$, then this means $g = 0$, so $f$ is constant.

c) We can not reach $1/6$ by adding non-negative integer multiples of $1/3$ and $1/2$, and $1$.

d) Since $G_k \in \mathbb{M}_k$ has the same zeros with multiplicites as any other $f \in \mathbb{M}_k$, $f/G_k \in \mathbb{M}_0$, so it is constant by b).

□

In particular, this shows that $G_4$ has its only root in $\mathbb{F}$ at $\rho$, which is of order 1, and $G_6$ has its only root at $i$, which is also of order 1.

The discriminant $\Delta$ lies in $\mathbb{M}_{12}$ and satisfies $\mathrm{ord}_\infty(\Delta) = 1$. The dimension formula now tells us that $\Delta$ has no other zeros in $\mathbb{F}$. For our calculation of the dimensions we need to link higher-weight entire modular forms to those already described. This is done by the following Lemma.

**Lemma.** *For even* $k \geq 12$ *we have* $\mathbb{S}_k = \Delta\mathbb{M}_{k-12}$. *Furthermore,* $\mathbb{M}_k = \mathbb{C}G_k^* \oplus \mathbb{S}_k$.

*Proof.* For the first statement, the inclusion "$\supseteq$" is clear. On the other hand, any function $f \in \mathbb{S}_k$, $k \geq 12$, has at least a zero of order 1 at $\infty$. Therefore, and since $\Delta$ has no zeros in $\mathbb{H}$, $f/\Delta$ is an entire modular form of weight $k - 12$.

For the second statement, notice that $G_k^*$ has no zero at $\infty$, so for some $f \in \mathbb{M}_k$, we can just subtract some scalar multiple of $G_k^*$ to get a cusp form of the same weight. □

This now gives us the possibility to calculate the dimensions of $\mathbb{M}_k$:

**Theorem.** *The spaces of entire modular forms,* $\mathbb{M}_k$ *for* $k$ *even, are finite-dimensional with dimensions given by*

$$\dim_{\mathbb{C}}(\mathbb{M}_k) = \begin{cases} \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor + 1, & k \not\equiv 2 \pmod{12} \end{cases}$$

*Proof.* This follows inductively from the Proposition earlier, as well as the relations

$$\dim(\mathbb{M}_k) = 1 + \dim(\mathbb{S}_k) \quad \text{and} \quad \dim(\mathbb{S}_{k+12}) = \dim(M_k),$$

which follow from the last Lemma. □

In the previous section we have seen that we can express $\Delta$ as a polynomial of $G_4^*$ and $G_6^*$. Also, we give the following proposition:

**Proposition.** *For any* $r, s \in \mathbb{N}$, *such that* $4r + 6s = k \geq 4$, *even, we have*

$$\mathbb{M}_k = \mathbb{C}G_4^{*r}G_6^{*s} \oplus \mathbb{S}_k.$$

*Proof.* Clearly, $G_4^{*r}G_6^{*s} \in \mathbb{M}_k$. Since $G_4^*$ and $G_6^*$ do not vanish at $\infty$, from any function we can subtract some complex multiple of $G_4^{*r}G_6^{*s}$ and get a cusp form. □

This eventually leads to our main result:

**Theorem.** *For $k \geq 4$ even, there is a decomposition*

$$\mathbb{M}_k = \bigoplus_{4r+6s=k} \mathbb{C}G_4^{*r}G_6^{*s}.$$

*This means that $\mathbb{M} = \bigoplus_k \mathbb{M}_k$ is freely generated as an algebra by $G_4^*$ and $G_6^*$. In other words, $\mathbb{M} = \mathbb{C}[G_4^*, G_6^*]$ and $G_4^*$ and $G_6^*$ algebraically independent.*

*Proof.* Due to the previous proposition and the relation $\mathbb{S}_k = \Delta\mathbb{M}_{k-12}$, by an inductive argument, every entire modular function can be written as a polynomial expression in $G_4^*$ and $G_6^*$. In terms of the graded evaluation homomorphism $\mathbb{C}[x^4, y^6] \to \mathbb{M}$; $x^4 \mapsto G_4^*$ and $y^6 \mapsto G_6^*$, we have surjectivity.

A standard combinatorial argument can show that the dimensions of the free, graded, commutative algebra $\mathbb{C}[x^4, y^6]$ is already exactly the same in each degree as those of $\mathbb{M}$. Therefore, the evaluation homomorphism is actually an isomorphism. □

# 5 L-functions

**Definition.** *Given an entire modular function of weight $k$ $f = \sum_{n=0}^{\infty} a_{f,n}q^n$ its L-function $L_f$ is defined by the following expression*

$$L_f(s) = \sum_{n=1}^{\infty} a_{f,n}n^{-s}.$$

*It is often normalized using the $\Gamma$-function $\Gamma(s) = \int_0^{\infty} e^{-y}y^{s-1}$ giving the definition*

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s).$$

**Remark.** *By standard analytic methods one can show that the sum representing the L-function converges absolutely and for $Re(s) > k$ even locally uniformly.*

**Theorem.** *Given an entire modular form of weight $k$ its L-function can be meromorphically extended to the whole of $\mathbb{C}$ only with possibly a pole at $s = k$ and residue*

$$\mathrm{Res}_{s=k}\, L_f(s) = \frac{(2\pi i)^k}{(k-1)!}a_f(0).$$

*Furthermore $L_f(0) = -a_f(0)$ and $L_f(-n) = n$ $\forall n \in \mathbb{N}$.*

*The functional equation*

$$\Lambda_f(k-s) = i^k\Lambda_f(s)$$

*holds.*

*Proof.* This follows from computing that

$$\Lambda_f(s) = \int_1^{\infty}\left(f(iy) - a_{f,0}(y^s + i^k y^{k-s})\frac{dy}{y}\right) + a_{f,0}(\frac{ik}{s-k} - \frac{1}{s}).$$

□

## References

[1] Max Koecher, Aloys Krieg, *Elliptische Funktionen und Modulformen*, Springer (2007).

[2] Markus Schwagenscheidt, *Vorlesung Modulformen*, lecture notes, Universität Köln (2019).