Elliptic functions

Quentin Roubaty and Lukas Keller

10.11.2020

Lattices in $\mathbb{C}$

Reminder on Meromorphic functions

We call a function f *meromorphic* if it is holomorphic everywhere but on a closed discrete set $D_f \subset \mathbb{C}$, where it has poles. Let $\mathcal{M}$ denote the set of meromorphic functions on $\mathbb{C}$. It can be checked that $\mathcal{M}$ is a field.

We call $\omega \in \mathbb{C}$ a period of a meromorphic function f if $D_f + \omega = D_f$ and $f(z+\omega) = f(z)$ for every $z \in \mathbb{C} \setminus D_f$. Let $\operatorname{Per} f$ denote the set of periods of f.

Lemma (fundamental-lemma)**.** For a non-constant $f \in \mathcal{M}$ one of three possible cases are true:

1. $\operatorname{Per} f = \{0\}$
2. There is a unique (up to sign) $\omega_f \in \mathbb{C} \setminus \{0\}$ such that $\operatorname{Per} f = \mathbb{Z}\omega_f$
3. There are linearly independent $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ (over $\mathbb{R}$) such that $\operatorname{Per} f = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\tau := \omega_1/\omega_2$ satisfies $\operatorname{Im}\tau > 0, |\operatorname{Re}\tau| \leq \frac{1}{2}$ and $|\tau| \geq 1$

The proof for this lemma can be found in [1].

Lattice

We will now introduce lattices and mention some of its properties, a lot more can be found in [1].

Definition (lattice)**.** A subset $\Omega \subset V$ of a real vector space is called a lattice if there is a basis $(\omega_1, \dots, \omega_n)$ of V such that $\Omega = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$.

Note that in the case 3 of the fundamental-lemma $\operatorname{Per} f$ is a lattice in the $\mathbb{R}$-vector space $\mathbb{C}$.

Proposition. Every lattice Ω in $\mathbb{C}$ is closed and discrete.

Proof. Let $\rho > 0$ and $M_\rho = \{\omega \in \Omega; |\omega| \leq \rho\}$. We can assume that $\Omega = \mathbb{Z}\tau + \mathbb{Z}, \tau = x + iy \in \mathbb{C}, y > 0$ (by replacing ρ by $\rho/|\omega_2|$).

Thus for $\omega = m\tau + n \in M_\rho$ the following two inequalities hold:

$$\rho^2 \geq |m\tau + n|^2 = (mx+n)^2 + m^2y^2 \geq m^2y^2$$

$$\rho \geq |mx + n| \geq |n| - |mx|$$

Then we can follow $|m| \leq \rho/y$ and $|n| \leq \rho(1 + |x|/y)$ which shows us that M_ρ is a finite set.

A sequence in Ω that converges in $\mathbb{C}$ is contained in some ball with radius ρ around 0. By finiteness of M_ρ we can then conclude that the limit point must be in $M_\rho \subset \Omega$. Thus Ω is closed. □

Definition (period parallelogram)**.** For a lattice Ω in $\mathbb{C}$ and $u \in \mathbb{C}$ we call

$$\diamond(u; \omega_1, \omega_2) := \{u + \alpha_1\omega_1 + \alpha_2\omega_2; 0 \leq \alpha_1 < 1, 0 \leq \alpha_2 < 1\}$$

a period parallelogram regarding ω_1, ω_2 with basepoint u. If $u = 0$ we can simplify the notation to $\diamond(\omega_1, \omega_2)$. Also one can just write P for any period parallelogram.

Proposition (A)**.** For every $z \in \mathbb{C}$ there is exactly one $\omega \in \Omega$ with $z + \omega \in P$.

Proof. Using the fact that ω_1, ω_2 form a basis of $\mathbb{C}$ we can pick $\xi_1, \xi_2 \in \mathbb{R}$ such that $z - u = \xi_1\omega_1 + \xi_2\omega_2$ and then

$$z - u + \lfloor \xi_1 \rfloor \omega_1 + \lfloor \xi_2 \rfloor \omega_2 \in \{\alpha_1\omega_1 + \alpha_2\omega_2; 0 \leq \alpha_1 < 1, 0 \leq \alpha_2 < 1\}$$

□

Elliptic functions

From now on we will always write $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for a lattice in $\mathbb{C}$.

Definition (elliptic function)**.** A meromorphic function f on $\mathbb{C}$ is called elliptic (or doubly periodic) regarding Ω if $\Omega \subset \operatorname{Per} f$, i.e. if

$$D_f + \omega = D_f \text{ and } f(z + \omega) = f(z)$$

for every $\omega \in \Omega$ and $z \in \mathbb{C} \setminus D_f$. We denote by $\mathcal{K}(\Omega)$ the set of elliptic functions regarding Ω.

The elliptic functions $\mathcal{K}(\Omega)$ form a subfield of the meromorphic functions on $\mathbb{C}$. The identity is clearly contained, the sum and product of two elliptic functions is again elliptic and also for any elliptic f also $\frac{1}{f}$ is again elliptic. (The identity theorem from complex analysis tells us that $\frac{1}{f}$ is again meromorphic as the zero set of $0 \neq f$ is closed and discrete.)

Liouville's theorems

We will now mention the four theorems of Liouville.

Theorem (A)**.** If $f \in \mathcal{K}(\Omega)$ is holomorphic then it is constant.

Proof. Let P be a periodic parallelogram. By compactness of $\bar{P}$, there is a $C > 0$ with $\forall z \in P : |f(z)| \leq C$. This bound can be extended to all of $\mathbb{C}$ by using proposition A: there is a $\omega_z \in \Omega$ such that $z + w \in P$ and thus $|f(z)| = |f(z + \omega)| \leq C$.

Then the result follows from another one of Liouville's theroems from complex analysis (every bounded holomorphic function must be constant). □

For the next two theorems let us first recall the residue and the order: $\operatorname{res}_c f := a_{-1}$, $\operatorname{ord}_c f := m$ where a_{-1} and m come from the laurent expansion of f:

$$f(z) = \sum_{n \geq m} a_n (z - c)^n, \quad a_m \neq 0, \quad m \in \mathbb{Z}$$

From here it can be shown that $\forall \omega \in \Omega : \operatorname{ord}_{c+\omega} f = \operatorname{ord}_c f$ and $\operatorname{res}_{c+\omega} f = \operatorname{res}_c f$ (Thus if c is a pole, so is $c + \omega$) using

$$f(z) = f(z - \omega) = \sum_{n \geq m} a_n (z - [c + \omega])^n$$

Theorem (B)**.** Let P be a periodic parallelogram and $f \in \mathcal{K}(\Omega)$, then it holds

$$\sum_{c \in P} \operatorname{res}_c f = 0$$

Proof. As f has only finitely many poles in P (D_f closed discrete and $\bar{P}$ compact), the sum is finite (almost all $\operatorname{res}_c = 0$). Using $\operatorname{res}_{c+\omega} f = \operatorname{res}_c f$ we see that the above sum is also independent of the choice of P. We can thus choose u such that there are no singularities on ∂P. Now one can use the residue theorem to show that the above sum must equal zero:

$$\begin{aligned}
\pm 2\pi i \sum_{c \in P} \operatorname{res}_c f &= \int_u^{u+\omega_1} f(z) dz + \int_{u+\omega_1}^{u+\omega_1+\omega_2} f(z) dz + \int_{u+\omega_1+\omega_2}^{u+\omega_2} f(z) dz + \int_{u+\omega_2}^{u} f(z) dz \\
&= \int_u^{u+\omega_1} \left(f(z) - f\left(z + \omega_2\right)\right) dz + \int_{u+\omega_2}^{u} \left(f(z) - f\left(z + \omega_1\right)\right) dz = 0
\end{aligned}$$

□

Theorem (C)**.** Let P be a periodic parallelogram and $f \in \mathcal{K}(\Omega)$ not constant, then we have $\forall w \in \mathbb{C}$:

$$\sum_{c \in P} \operatorname{ord}_c (f - w) = 0$$

And f assumes every value in $\mathbb{C}$

Proof. Define $g(z) := \frac{f'(z)}{f(z) - w}$. This function is again elliptic regarding Ω. (This follows from the fact that the derivative of a periodic function is again periodic and because $\mathcal{K}(\Omega)$ is a field). Now using that $\operatorname{res}_c g = \operatorname{ord}_c (f - w)$ (see lemma below) we can apply Thm. B to g to get the first result.

Because f is non-constant Thm. A implies that $f - w$ has poles, then because of $\operatorname{ord}_{c+\omega}(f - w) = \operatorname{ord}_c(f - w)$, there also must exist a pole in P. Thus $f - w$ must also have a root in P (because the above sum is zero). As w was arbitrary, f assumes every value in $\mathbb{C}$

□

Note that this proof also implies that (counting with multiplicities):

$$\begin{aligned}
0 < \text{number of poles of } f \text{ in } P &= \text{number of poles of } f - w \text{ in } P \\
&= \text{number of roots of } f - w \text{ in } P \\
&= \text{number of roots of } f \text{ in } P \text{ , (pick } w = 0)
\end{aligned}$$

Lemma. For g as above we have $\operatorname{res}_c g = \operatorname{ord}_c(f-w)$

Proof. As above, $f-w$ has finitely many zeros $z_1,\dots,z_r$ (with multiplicity $\alpha_1,\dots,\alpha_r$) and poles $\omega_1,\dots,\omega_s$ (with multiplicity $\beta_1,\dots,\beta_s$) in P.

For every z_i there is a holomorphic function $\hat{f}$ with no roots such that $f(z)-w=(z-z_i)^{\alpha_i}\hat{f}(z)$ on some neighbourhood. Then:

$$f'(z) = \alpha_i(z-z_i)^{\alpha_i-1}\hat{f} + (z-z_i)^{\alpha_i}\hat{f}'(z)$$

$$g(z) = \frac{\alpha_i(z-z_i)^{\alpha_i-1}\hat{f} + (z-z_i)^{\alpha_i}\hat{f}'(z)}{(z-z_i)^{\alpha_i}\hat{f}(z)} = \frac{\alpha_i}{z-z_i} + \frac{\hat{f}'(z)}{\hat{f}(z)}$$

Thus $\operatorname{res}_{z_i}(g) = \alpha_i = \operatorname{ord}_{z_i}(f-w)$

Similarly for every ω_i there is a holomorphic function $\hat{f}$ with no roots such that $f(z)-w = \frac{1}{(z-\omega_i)^{\beta_i}}\hat{f}(z)$ on some neighbourhood. Then:

$$f'(z) = \frac{-\beta_i}{(z-\omega_i)^{\beta_i+1}}\hat{f} + \frac{1}{(z-\omega_i)^{\beta_i}\hat{f}'(z)}$$

$$g(z) = \frac{-\beta_i}{z-\omega_i} + \frac{\hat{f}'(z)}{\hat{f}(z)}$$

Thus $\operatorname{res}_{\omega_i}(g) = -\beta_i = \operatorname{ord}_{\omega_i}(f-w)$.

All the other points in $c\in P$ have $\operatorname{res}_c(g) = 0 = \operatorname{ord}_c(f-w)$ $\square$

Theorem (D)**.** Let P be a periodic parallelogram and $0\neq f\in\mathcal{K}(\Omega)$, then $\sum_{c\in P}(\operatorname{ord}_c f)\cdot c\in\Omega$

Proof. Let $c\in P$ and $m=\operatorname{ord}_c f$, as before we can write $f(z)=(z-c)^m\hat{f}(z)$ for holomorphic $\hat{f}$ with no roots in a neighbourhood of z.

$$\frac{f'}{f} = \frac{m(z-c)^{m-1}\hat{f} + (z-c)^m\hat{f}'(z)}{(z-c)^m\hat{f}(z)} = \frac{m}{z-c} + \frac{\hat{f}'(z)}{\hat{f}(z)}$$

Thus $\operatorname{ord}_c f = \operatorname{res}_c \frac{f'}{f}$

Then using the residue formula for simple poles we have:

$$\operatorname{res}_c\left(\frac{f'}{f}z\right) = \operatorname{res}_c\left(\frac{mz}{z-c}\right) = \lim_{z\to c}(z-c)\frac{mz}{z-c} = mc$$

Thus combining both results gives us

$$c\cdot\operatorname{ord}_c f = c\cdot\operatorname{res}_c\frac{f'}{f} = cm = \operatorname{res}_c\left(\frac{f'}{f}z\right)$$

Now we can apply the residue theorem like we did in Thm. B:

$$2\pi i \sum_{c\in P} (\mathrm{ord}_c\, f)\cdot c = 2\pi i \sum_{c\in P}\left(\mathrm{res}_c\left(\frac{f'}{f}z\right)\right) = \int_{\partial P} z\cdot \frac{f'(z)}{f(z)}dz$$
$$= \pm\left(\int_u^{u+\omega_1} z\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z+\omega_2)}{f(z+\omega_2)}dz + \int_{u+\omega_2}^{u} z\frac{f'(z)}{f(z)} - (z+\omega_1)\frac{f'(z+\omega_1)}{f(z+\omega_1)}dz\right)$$
$$= \pm\left(\omega_1\int_u^{u+\omega_2}\frac{f'(z)}{f(z)}dz - \omega_2\int_u^{u+\omega_1}\frac{f'(z)}{f(z)}dz\right)$$
$$= \pm\left(\omega_1\int_0^1\frac{f'(u+t\omega_2)}{f(u+t\omega_2)}\omega_2 dt - \omega_2\int_0^1\frac{f'(u+t\omega_1)}{f(u+t\omega_1)}\omega_1 dt\right)$$
$$= \pm\left(\omega_1(\log f(u+\omega_2) - \log f(u) + 2\pi i k) - \omega_2(\log f(u+\omega_1) - \log f(u) - 2\pi i\tilde{k}\right)$$
$$= \pm\,\omega_1 2\pi i k + \omega_2 2\pi i\tilde{k}$$

Thus the result follows. □

Weierstrass $\wp$ function

Construction of the $\wp$-function

The definition of the Weierstrass $\wp$-function is coupled to the following theorem:

Theorem (Construction theorem of $\wp$-function)**.** The series

$$\wp(z) := \wp_\Omega(z) := z^{-2} + \sum_{0\neq\omega\in\Omega}\left((z-\omega)^{-2} - \omega^{-2}\right), z\in\mathbb{C}\backslash\Omega$$

converges in every compact set in $\mathbb{C}\setminus\Omega$ uniformly and absolutely. The function $\wp$, called the Weierstass $\wp$-function, is an even elliptic function regarding Ω, it has poles of order 2 on Ω with residue 0 and is holomorphic on $\mathbb{C}\setminus\Omega$. The Laurent series at 0 is

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + \mathcal{O}(z^4).$$

Corollary. (i) $\wp$ is even

(ii) $\wp'$ is odd, has poles of order 3 on Ω and is holomorphic everywhere else

(iii) $\omega\in\Omega\not\ni\frac{\omega}{2}\Leftrightarrow\frac{\omega}{2}$ is a simple root of $\wp'$

We will prove (iii):

Proof (corollary). Because $\wp'$ is odd we have

$$\wp'(z+\omega) = \wp'(z) = -\wp'(-z)$$

Note that $\frac{\omega}{2}\notin\Omega\Rightarrow\frac{\omega}{2}$ is no pole of $\wp$ and $\wp'$. Thus we can let $z=-\frac{\omega}{2}$:

$$\wp'\left(\frac{\omega}{2}\right) = -\wp'\left(\frac{\omega}{2}\right)\Rightarrow\wp'\left(\frac{\omega}{2}\right) = 0$$

Let ω_1,ω_2 be a base of Ω, $P=\diamond(\omega_1,\omega_2)$. Then $\wp'$ has roots $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_1+\omega_2}{2}$.

By the laurent expansion, $\wp'$ only has one pole of order 3 at 0 in P, by Thm. C also the number of roots in P must equal 3. Thus those roots are all of order 1 and the list is complete.

If z is any root of $\wp'$, there is a $\omega' \in \Omega$ with $z - \omega' \in P$. But then $z - \omega'$ is one of the roots above and thus $z = \frac{\omega_i}{2} + \omega'$ or $z = \frac{\omega_1+\omega_2}{2} + \omega'$, i.e. z has the desired form. □

Proof (theorem). We only show the absolute convergence. Let $K \subset \mathbb{C} \setminus \Omega$ be compact, we consider $\rho > 0$ such that $K \subset \mathcal{B}_\rho(0)$. Then all $\omega \in \Omega$ with $|\omega| \geq \rho + 1$ are satisfying

$$\begin{aligned}\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{2z\omega - z^2}{\omega^2(z-\omega)^2}\right| &= \left|\frac{z}{\omega^3}\right| \left|\frac{2 - \frac{z}{\omega}}{(1-\frac{z}{\omega})^2}\right| \\ &\leq \frac{\rho}{|\omega|^3} \frac{3}{(1-\frac{\rho}{\rho+1})^2} = 3\rho(\rho+1)^2 \frac{1}{|\omega|^3}\end{aligned}$$

and thus $\forall z \in K$

$$\begin{aligned}|\wp(z)| &\leq \left|\frac{1}{z^2}\right| + \sum_{0 \neq \omega \in \Omega} \left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| \\ &\leq \left|\frac{1}{z^2}\right| + \sum_{0<|\omega|<\rho+1} \left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| + 3\rho(\rho+1)^2 \frac{1}{|\omega|^3}\end{aligned}$$

Again using compactness of K we now that both the terms

$$\left|\frac{1}{z^2}\right| \quad \text{and} \quad \sum_{0<|\omega|<\rho+1} \left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right|$$

are bounded by some constants which does not depend on z. Thus the absolute convergence is a consequence of the following lemma. □

Lemma. Given a lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ and some real $\alpha \in \mathbb{R}$. The series $\sum\limits_{0\neq\omega\in\Omega} \frac{1}{|\omega|^\alpha}$ is convergent if and only if $\alpha > 2$.

Proof. The proof of this lemma ask for those three definitions:

$$\begin{aligned} \delta(\Omega) &= \mathrm{Sup}\big\{|z-w| \bigm| z,w \in \Diamond(\omega_1,\omega_2)\big\} \\ A_\rho(\Omega) &= \#\big\{\omega \in \Omega \bigm| |\omega| \leq \rho\big\} \\ \mathrm{vol}(\Omega) &= \mathrm{vol}(\Diamond(\omega_1,\omega_2)) \end{aligned}$$

and for a well-known result about lattices which states that $\forall\rho \geq \delta$:

$$\frac{\pi}{\mathrm{vol}(\Omega)}(\rho-\delta)^2 \leq A_\rho \leq \frac{\pi}{\mathrm{vol}(\Omega)}(\rho+\delta)^2$$

We only prove the "if"-part.
Let denote $E_N = \{\omega \in \Omega \mid 0 < |\omega| < N\}$ for $N \in \mathbb{N}$. We then have that

$$\sum_{0\neq\omega\in\Omega} \frac{1}{|\omega|^\alpha} = \lim_{N\to\infty}\left(\sum_{\omega\in E_N} \frac{1}{|\omega|^\alpha}\right) \leq c_1 + \lim_{N\to\infty}\left(\sum_{\substack{n\in\mathbb{N} \\ \delta<n<N}} (A_{n+1}-A_n)\frac{1}{n^\alpha}\right)$$

where $c_1 := \sum\limits_{0<|\omega|\leq\delta} \frac{1}{|\omega|^\alpha}$ is a constant. Now using the upper and lower bounds of A_n and A_{n+1} we get that

$$\begin{aligned} A_{n+1} - A_n &\leq \frac{\pi}{\mathrm{vol}(\Omega)}\big((n+1+\delta)^2 - (n-\delta)^2\big) \\ &= \frac{\pi}{\mathrm{vol}(\Omega)}((4\delta+2)n + 2\delta) =: c_2 n + c_3 \end{aligned}$$

and thus

$$\begin{aligned} \sum_{0\neq\omega\in\Omega} \frac{1}{|\omega|^\alpha} &\leq c_1 + \lim_{N\to\infty}\left(\sum_{\substack{n\in\mathbb{N} \\ \delta<n<N}} \frac{c_2 n + c_3}{n^\alpha}\right) \\ &\leq c_1 + c_2\sum_{n=1}^{\infty}\frac{1}{n^{\alpha-1}} + c_3\sum_{n=1}^{\infty}\frac{1}{n^\alpha} \end{aligned}$$

This will converges since both α an $\alpha - 1$ are strictly greater than 1. The "only if"-part use the same kind of trick to compare the series with the harmonic one and shows that it diverges. □

The differential equation

We now want to show that $\wp$ satisfies some differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

where g_2, g_3 are constant depending only on Ω.
We begin by studying the Laurent-series of $\wp$ at 0. We want to write $\wp(z)$ in the form $\frac{1}{z^2} + a_2z^2 + a_4z^4 + ...$ in some open neighborhood of 0. The trick that will be used is the following:

$$\forall\, 0 < |z| < 1 \quad : \quad \frac{1}{(1-z)^2} = \frac{d}{dz}(\frac{1}{1-z}) = \frac{d}{dz}(\sum_{m=1}^{\infty} z^m) = \sum_{m=1}^{\infty} mz^{m-1}$$

Since $\Omega \in \mathbb{C}$ is discrete, we can take $r > 0$ such that $\Omega \cap \mathcal{B}_r(0) = 0$. On $\mathcal{B}_r(0)$ we then have

$$\begin{aligned}\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \frac{1}{\omega^2}\left(\frac{1}{(1-\frac{z}{\omega})^2} - 1\right) = \frac{1}{\omega^2}\left(\sum_{m=1}^{\infty} m\left(\frac{z}{\omega}\right)^{m-1} - 1\right) \\ &= \frac{1}{\omega^2}\left(\sum_{m=2}^{\infty} m\left(\frac{z}{\omega}\right)^{m-1}\right) = \sum_{m=2}^{\infty} mz^{m-1}\frac{1}{\omega^{m+1}}\end{aligned}$$

and thus

$$\sum_{0\neq\omega\in\Omega}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) = \sum_{m=2}^{\infty} mz^{m-1}\left(\sum_{0\neq\omega\in\Omega}\frac{1}{\omega^{m+1}}\right) =: \sum_{m=2}^{\infty} mz^{m-1}G_{m+1}$$

where those $G_k = \sum\limits_{0\neq\omega\in\Omega} \frac{1}{\omega^k}$ are called the Eisenstein-series of the lattice Ω. Those series are absolute convergent for $k \geq 3$ by the Lemma above.
Moreover, $G_k = 0$ for all odd value of $k \geq 3$ by symmetry of the lattice. Indeed

$$-G_k = -\sum_{0\neq\omega\in\Omega}\frac{1}{\omega^k} = \sum_{0\neq\omega\in\Omega}\frac{1}{(-\omega)^k} = \sum_{0\neq\omega\in\Omega}\frac{1}{\omega^k} = G_k$$

So our Laurent-series at 0 is given by

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)z^{2n}G_{2n+2} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \mathcal{O}(z^6)\end{aligned}$$

From the Laurent-series of $\wp$ we can compute the Laurent-series of the following functions at 0:

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \mathcal{O}(z^6) \\ \wp(z)^2 &= \frac{1}{z^4} + 6G_4 + 10G_6z^2 + \mathcal{O}(z^4) \\ \wp(z)^3 &= \frac{1}{z^6} + 9G_4\frac{1}{z^2} + 15G_6 + \mathcal{O}(z^2) \\ \wp'(z) &= -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + \mathcal{O}(z^5) \\ \wp'(z)^2 &= \frac{4}{z^6} - 24G_4\frac{1}{z^2} - 80G_6 + \mathcal{O}(z^2)\end{aligned}$$

and this give us the following equation:

$$f(z) := \wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = \mathcal{O}(z^2)$$

Since $\mathcal{K}(\Omega)$ is a field, we get that $f \in \mathcal{K}(\Omega)$. Moreover the equation tell us that f is holomorphic at 0 (and thus everywhere). Thus we get that f is constant by the first Liouville's theorem. Again by the equation we conclude that $f \equiv 0$. So we get our differential equation

$$\begin{aligned}\wp'^2 &= 4\wp^3 - 60G_4\wp - 140G_6 \\ &=: 4\wp^3 - g_2\wp - g_3\end{aligned}$$

Our next goal is to show that $\wp$ is essentially the only non-trivial meromorphic solution of this differential equation. That is, we want to show the following corollary.

Corollary. Given $D \subset \mathbb{C}$ a domain and f a non-constant meromorphic function on D satisfying

$$f'^2 = 4f^3 - g_2 f - g_3$$

Then there exists $z_0 \in \mathbb{C}$ such that $f(z) = \wp(z + z_0)$ on D.

Remark. This corollary gives us that any meromorphic solution of this differential equation is necessarily elliptic with respect to Ω. This also gives us that the lattice Ω is uniquely defined by g_2, g_3 or by G_4, G_6.

Proof. The idea is to find some open and simply-connected set $U \subset D$ on which we can transform our differential equation in something of the form

$$f'(z) = g(z, f(z))$$

where g is continuous and Lipschitz-continuous in the second variable and then to use the existence and uniqueness of a solution on U.
So let pick a point $u \in D$ with an open simply-connected neighborhood $u \in U \subset D$ such that f is holomorphic and f' is non-vanishing on U. Moreover, we can take U small enough such that

$$f' = \sqrt{4f^3 - g_2 f - g_3}$$

on U for some branch of the square root.
Now by third Liouville's Theorem we know that there exists some $z_1 \in \mathbb{C}$ with $\wp(z_1) = f(u)$. Since $\wp$ is even we also have $\wp(-z_1) = f(u)$. And cause $\wp'$ is odd we can take w.l.o.g that $\wp'(z_1) = f'(u)$ (otherwise we just replace z_1 by $-z_1$). Now let define $z_0 := z_1 - u$. We get that both

$$f(z) \quad \text{and} \quad g(z) := \wp(z + z_0)$$

are satisfying our new initial value problem on U. So by uniqueness of the solution, we get that both are equal on U. Finally we conclude that $f \equiv g$ on D by the identity theorem of meromorphic functions. $\square$

References

[1] Koecher, Krieg, Elliptische Funktionen und Modulformen